

FINITE DIMENSIONAL HOPF ALGEBRAS OVER KAC-PALJUTKIN ALGEBRA H_8

YUXING SHI

ABSTRACT. Let H_8 be the neither commutative nor cocommutative semisimple eight dimensional Hopf algebra, which is also called Kac-Paljutkin algebra [KP66]. All simple Yetter-Drinfel'd modules over H_8 are given. As for simple objects and direct sums of two simple objects in ${}^{H_8}_{H_8}\mathcal{YD}$, we calculated dimensions for the corresponding Nichols algebras, except four semisimple cases which are generally difficult. Under the assumption that the four undetermined Nichols algebras are all infinite dimensional, we determine all the finite dimensional Nichols algebras over H_8 . It turns out that the already known finite dimensional Nichols algebras are all diagonal type. In fact, they are Cartan types A_1 , A_2 , $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$. By the way, we calculate Gelfand-Kirillov dimensions for some Nichols algebras. As an application, we obtain five families of new finite dimensional Hopf algebras over H_8 according to the lifting method.

1. INTRODUCTION

Let \mathbb{K} be an algebraically closed field of characteristic zero. The question of classification of all Hopf algebras over \mathbb{K} of a given dimension up to isomorphism was posed by Kaplansky in 1975 [Kap75]. Some progress has been made but, in general, it is a difficult question for lack of standard methods. One breakthrough is the so-called *Lifting Method* [AS98] introduced by Andruskiewitsch and Schneider in 1998, under the assumption that the coradical is a Hopf subalgebra.

We describe the procedure for the lifting method briefly. Let H be a Hopf algebra whose coradical H_0 is a Hopf subalgebra. It is well-known that the associated graded Hopf algebra of H is isomorphic to $R\#H_0$ where $R = \bigoplus_{n \in \mathbb{N}_0} R(n)$ is a braided Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of Yetter-Drinfeld modules over H_0 . $\#$ stands for the Radford biproduct or *bosonization* of R with H_0 . As explained in [AS02], to classify finite-dimensional Hopf algebras H whose coradical is isomorphic to H_0 we have to deal with the following questions:

1991 *Mathematics Subject Classification.* Primary 17B37, 81R50; Secondary 17B35.

Keywords: Nichols algebra; Hopf algebra; Gelfand-Kirillov dimension; Kac-Paljutkin algebra.

- (a) Determine all Yetter-Drinfel'd modules V over H_0 such that the Nichols algebra $\mathfrak{B}(V)$ has finite dimension; find an efficient set of relations for $\mathfrak{B}(V)$.
- (b) If $R = \bigoplus_{n \in \mathbb{N}_0} R(n)$ is a finite-dimensional Hopf algebra in ${}^{H_0}_{H_0} \mathcal{YD}$ with $V = R(1)$, decide if $R \simeq \mathfrak{B}(V)$. Here $V = R(1)$ is a braided vector space called the *infinitesimal braiding*.
- (c) Given V as in (a), classify all H such that $\text{gr } H \simeq \mathfrak{B}(V) \# H_0$ (lifting).

The lifting method was extensively used in the classification of finite dimensional pointed Hopf algebras such as [AS10], [AHS10], [GGI11][FG11], [AFGV11], [AFGV10], [ACG16], [ACG15] and so on. It is also effective to study finite-dimensional copointed Hopf algebras [AV11], [GIV14], [FAGM16]. We note that there are very few classification results on finite-dimensional Hopf algebras whose coradical is a Hopf subalgebra but not a group algebra and the dual of a group algebra, two exceptions being [CDMM04, AM16].

Here we would like to initiate a project for the study of Hopf algebras whose coradicals are low-dimensional neither commutative nor cocommutative semisimple Hopf algebras by running procedures of the lifting method. One important step is to study the Nichols algebras over those low-dimensional semisimple Hopf algebras. Nichols algebras were studied first by Nichols [Nic78]. These are connected graded braided Hopf algebras [And02] generated by primitive elements, and all primitive elements are of degree one. In the past decades, the study of Nichols algebras was mainly focused on group algebras and which were finite dimensional, for those Nichols algebras were essential ingredients of the classification of finite-dimensional pointed Hopf algebras. Under the assumption that the base field has characteristic 0, the classification of finite-dimensional Nichols algebras over abelian groups was completely solved in [Hec06, Hec09] by using Lie theoretic structures, and the result of the classification played an important role later in the significant work [AS10]. The problem of classifying finite-dimensional Nichols algebras over non-abelian groups is difficult in general for lack of systematic method, related works please refer to [AHS10], [FGV10], [GHV11], [HLV12] [HS10], [HV14], [HV15], etc.

In this paper, we mainly focus on Kac-Paljutkin algebra H_8 . The structure of our paper is as follows. In Section 2, we recall the fundamental notions related to Yetter-Drinfel'd modules, Nichols algebras and Gelfand-Kirillov dimension. In section 3, we construct all the simple left Yetter-Drinfel'd modules over H_8 according to Radford's method. In section 4, we get all the possible finite dimensional Nichols algebras from Yetter-Drinfel'd modules over H_8 under the assumption that the four undetermined cases over semisimple modules for which are difficult for us at the moment. It turns out that all the already known finite dimensional Nichols algebras are Cartan types A_1 , A_2 , $A_2 \times A_2$, $A_1 \times \cdots \times A_1$, and $A_1 \times \cdots \times A_1 \times A_2$. Here is our first main result.

Theorem A. *Suppose*

$$\dim \mathfrak{B}(M\langle(xy, x)\rangle \oplus W^{b_1, -1}) = \infty = \dim \mathfrak{B}(M\langle(y, xy)\rangle \oplus W^{b_1, -1})$$

holds for $b_1 = \pm 1$. If $M \in {}_{H_8}^{H_8}\mathcal{YD}$ such that $\dim \mathfrak{B}(M) < \infty$, then M is isomorphic either to one of the following Yetter-Drinfel'd modules

- (1) $\Omega_1(n_1, n_2, n_3, n_4) \triangleq \bigoplus_{j=1}^4 M\langle 1|V_1(b_j), g_j \rangle^{\oplus n_j}$ with $\sum_{j=1}^4 n_j \geq 1$, $(b_1, g_1) = (i, x)$, $(b_2, g_2) = (-i, x)$, $(b_3, g_3) = (i, y)$, $(b_4, g_4) = (-i, y)$, the infinitesimal braiding is type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2+n_3+n_4}$.
- (2) $\Omega_2(n_1, n_2) \triangleq M\langle i, x \rangle^{\oplus n_1} \oplus M\langle -i, x \rangle^{\oplus n_2} \oplus M\langle(xy, x)\rangle$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.
- (3) $\Omega_3(n_1, n_2) \triangleq M\langle i, y \rangle^{\oplus n_1} \oplus M\langle -i, y \rangle^{\oplus n_2} \oplus M\langle(y, xy)\rangle$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.
- (4) $\Omega_4(n_1, n_2) \triangleq M\langle i, x \rangle^{\oplus n_1} \oplus M\langle i, y \rangle^{\oplus n_2} \oplus W^{1, -1}$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.
- (5) $\Omega_5(n_1, n_2) \triangleq M\langle -i, x \rangle^{\oplus n_1} \oplus M\langle -i, y \rangle^{\oplus n_2} \oplus W^{-1, -1}$, $n_1 + n_2 \geq 0$, the infinitesimal braiding is type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$.
- (6) $\Omega_6 \triangleq M\langle(xy, x)\rangle \oplus M\langle(y, xy)\rangle$, the infinitesimal braiding is type $A_2 \times A_2$.
- (7) $\Omega_7 \triangleq W^{1, -1} \oplus W^{-1, -1}$, the infinitesimal braiding is type $A_2 \times A_2$.

Remark 1.1. We point out which of the Yetter-Drinfeld modules have a principal realization and which not, since the liftings are known when there is a principal realization and not otherwise [AAI]. Let (h) and (δ_h) be a dual basis of H_8 and H_8^* , and $b \in \{\pm 1, \pm i\}$, then

$$\chi_b := \delta_1 + \delta_{xy} + b(\delta_x + \delta_y) + b^2(\delta_z + \delta_{zy}) + b^3(\delta_{zx} + \delta_{zy}) \in \text{Alg}(H_8, \mathbb{K}).$$

(g, χ_b) is a YD-pair [AAI⁺14] and $\mathbb{K}_g^{\chi_b} \simeq M\langle b, g \rangle \cdot M\langle(g_1, g_2)\rangle$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ and $W^{b_1, -1}$ for $b_1 = \pm 1$ don't have a principal realization [AAI, Subsection 2.2], since $\mathbb{K}_{g_1}^{\chi_{b_1}} \oplus \mathbb{K}_{g_2}^{\chi_{b_2}}$ is of type $A_1 \times A_1$ for $(b_1, g_1), (b_2, g_2) \in \{(\pm i, x), (\pm i, y)\}$. So only $\Omega_1(n_1, n_2, n_3, n_4)$ has a principal realization.

In section 5, according to the lifting method, we give a classification for finite-dimensional Hopf algebras over H_8 such that their infinitesimal braidings are isomorphic to those Yetter-Drinfel'd modules listed in Theorem A. Here is the second main result.

Theorem B. Suppose H is a finite-dimensional Hopf algebra over H_8 such that its infinitesimal braiding is isomorphic to one of the Yetter-Drinfel'd modules listed in Theorem A, then H is isomorphic either to

- (1) $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$, see Definition 5.4;
- (2) $\mathfrak{B}[\Omega_2(n_1, n_2)]\#H_8$, see Proposition 5.10;
- (3) $\mathfrak{A}_4(n_1, n_2; I_4)$, see Definition 5.19;
- (4) $\mathfrak{A}_6(\lambda)$, see Definition 5.11;
- (5) $\mathfrak{A}_7(I_7)$, see Definition 5.16.

$\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ comprises two nonisomorphic nonpointed self-dual Hopf algebras of dimension 16 with coradical H_8 described in [CDMM04] as special cases. Except $\mathfrak{B}[\Omega_2(n_1, n_2)]\#H_8$, the remainder four families of Hopf algebras contain non-trial lifting relations.

2. PRELIMINARIES

2.1. Conventions. Let H be a Hopf algebra over \mathbb{K} , with antipode S . We will use Sweedler's notation $\Delta(h) = h_{(1)} \otimes h_{(2)}$ for the comultiplication [Mon93]. Let ${}^H_H\mathcal{YD}$ be the category of left Yetter-Drinfel'd modules over H . That is to say that if M is an object of ${}^H_H\mathcal{YD}$ if and only if there exists an action \cdot such that (M, \cdot) is a left H -module and a coaction ρ such that (M, ρ) is a left H -comodule, subject to the following compatibility condition:

$$(2.1) \quad \rho(h \cdot m) = h_{(1)}m_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot m_{(0)}, \forall m \in M, h \in H,$$

where $\rho(m) = m_{(-1)} \otimes m_{(0)}$. It is a braided monoidal category. The braiding $c \in \text{End}_{\mathbb{K}}(M \otimes M)$ of M is defined by $c(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$, and the inverse braiding is defined by $c^{-1}(v \otimes w) = w_{(0)} \otimes (S^{-1}(w_{(-1)}) \cdot v)$.

Definition 2.1. [AS02, Definition. 2.1] Let H be a Hopf algebra and $V \in {}^H_H\mathcal{YD}$. A braided \mathbb{N} -graded Hopf algebra $R = \bigoplus_{n \geq 0} R(n) \in {}^H_H\mathcal{YD}$ is called the *Nichols algebra* of V if

- (i) $\mathbb{K} \simeq R(0)$, $V \simeq R(1) \in {}^H_H\mathcal{YD}$,
- (ii) $R(1) = \mathcal{P}(R) = \{r \in R \mid \Delta_R(r) = r \otimes 1 + 1 \otimes r\}$.
- (iii) R is generated as an algebra by $R(1)$.

In this case, R is denoted by $\mathfrak{B}(V) = \bigoplus_{n \geq 0} \mathfrak{B}^n(V)$.

Remark 2.2. The Nichols algebra $\mathfrak{B}(V)$ is completely determined by the braiding. Let $\mathfrak{B}(M)$ denote the Nichols algebra generated by $M \in {}^H_H\mathcal{YD}$. More precisely, as proved in [Sch96]

and noted in [AS02],

$$\mathfrak{B}(M) = K \oplus M \oplus \bigoplus_{n=2}^{\infty} M^{\otimes n} / \ker \mathfrak{S}_n = T(M) / \ker \mathfrak{S},$$

where $\mathfrak{S}_{n,1} \in \text{End}_{\mathbb{K}}(M^{\otimes(n+1)})$, $\mathfrak{S}_n \in \text{End}_{\mathbb{K}}(M^{\otimes n})$,

$$\mathfrak{S}_{n,1} := \text{id} + c_n + c_n c_{n-1} + \cdots + c_n c_{n-1} \cdots c_1 = \text{id} + c_n \mathfrak{S}_{n-1,1}$$

$$\mathfrak{S}_1 := \text{id}, \quad \mathfrak{S}_2 := \text{id} + c, \quad \mathfrak{S}_n := (\mathfrak{S}_{n-1} \otimes \text{id}) \mathfrak{S}_{n-1,1}.$$

Lemma 2.3. ([Gn00, Theorem 2.2], [AAH16, Remark 1.4]) *Let $M_1, M_2 \in {}^H_H\mathcal{YD}$ be both finite dimensional and assume $c_{M_1, M_2} c_{M_2, M_1} = \text{id}_{M_2 \otimes M_1}$. Then $\mathfrak{B}(M_1 \oplus M_2) \simeq \mathfrak{B}(M_1) \otimes \mathfrak{B}(M_2)$ as graded vector spaces and $\text{GKdim } \mathfrak{B}(M_1 \oplus M_2) = \text{GKdim } \mathfrak{B}(M_1) + \text{GKdim } \mathfrak{B}(M_2)$.*

Proposition 2.4. ([Rad85, Radford biproduct]) *H is a Hopf algebra. Let $A \in {}^H_H\mathcal{YD}$ be a braided Hopf algebra. Then $A \# H$ is a Hopf algebra.*

$$(2.2) \quad (a \# h)(a' \# h') = \sum a(h_{(1)} \cdot a') \# h_{(2)} h', \quad a, a' \in A, h, h' \in H$$

$$(2.3) \quad \Delta(a \# h) = \sum [a_{(1)} \# (a_{(2)})_{(-1)} h_{(1)}] \otimes [(a_{(2)})_{(0)} \# h_{(2)}]$$

$$(2.4) \quad S(a \# h) = \sum (1 \# S_H(h) S_H(a_{(-1)})) (S_A(a_{(0)}) \# 1)$$

The map $\iota : H \rightarrow A \# H$ given by $\iota(h) = 1 \# h$ for all $h \in H$ is an injective Hopf algebra map, and the map $\pi : A \# H \rightarrow H$ given by $\pi(a \# h) = \varepsilon_A(a) h$ for all $a \in A, h \in H$ is a surjective Hopf algebra map such that $\pi \circ \iota = \text{id}_H$. Moreover, it holds that $A = (A \# H)^{\text{co}\pi}$.

Conversely, let B be a Hopf algebra with bijective antipode and $\pi : B \rightarrow H$ a Hopf algebra epimorphism admitting a Hopf algebra section $\iota : H \rightarrow B$ such that $\pi \circ \iota = \text{id}_H$. Then $A = B^{\text{co}\pi}$ is a braided Hopf algebra in ${}^H_H\mathcal{YD}$ and $B \simeq A \# H$ as Hopf algebras.

2.2. GK-dimension. Let A be a finitely generated algebra over a field \mathbb{K} , and assume a_1, \dots, a_m generate A . Set V to be the span of a_1, \dots, a_m , and denote V^n the span of all monomials in the a_i 's of length n . As a_i 's generate A one has $A = \bigcup_{k=0}^{\infty} A_k$ where $A_k = \mathbb{K} + V + V^2 + \cdots + V^k$. The function $d_V(n) = \dim A_n$ is the growth function of A . The *Gelfand-Kirillov dimension* of a \mathbb{K} -algebra A is $\text{GKdim } A = \overline{\lim} \log_n d_V(n)$. $\text{GKdim } A$ does not depend on the choice of V . Suppose that $\text{GKdim } A < \infty$. We say that a finite-dimensional subspace $V \subseteq A$ is *GK-deterministic* if

$$(2.5) \quad \text{GKdim } A = \lim_{n \rightarrow \infty} \log_n \dim \sum_{0 \leq j \leq n} V^j.$$

	1	x	y	z
$\tau_1 = \text{id}$	1	x	y	z
τ_2	1	x	y	xyz
τ_3	1	y	x	$\frac{1}{2}(z + xz + yz - xyz)$
τ_4	1	y	x	$\frac{1}{2}(-z + xz + yz + xyz)$

TABLE 1. Automorphisms of H_8

Clearly, if V is a GK -deterministic subspace of A , then any finite-dimensional subspace of A containing V is GK -deterministic. Let A and B be two algebras. Then

$$(2.6) \quad \text{GKdim}(A \otimes B) \leq \text{GKdim } A + \text{GKdim } B,$$

but the equality does not hold in general. For instance, it does hold when A or B has a GK -deterministic subspace, see [KL00, Proposition 3.11]. The Gelfand-Kirillov dimension is a useful tool in Ring theory and Hopf algebraic theories. We shall not discuss in detail its importance but we refer the reader to [KL00] as a basic reference and [WLD16, WZZ13, BZ10, AAH16] for additional informations related with Hopf algebras.

3. SIMPLE YETTER-DRINFEL'D MODULES OF H_8

Recall that the neither commutative nor cocommutative semisimple 8-dimensional Hopf algebra H_8 in [Mas95] is constructed as an extension of $\mathbb{K}[C_2 \times C_2]$ by $\mathbb{K}[C_2]$. A basis for H_8 is given by $\{1, x, y, xy = yx, z, xz, yz, xyz\}$ with the relations

$$x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad zx = yz, \quad zy = xz.$$

The coalgebra structure and the antipode are defined by

$$\begin{aligned} \Delta(x) &= x \otimes x, & \Delta(y) &= y \otimes y, & \varepsilon(x) &= \varepsilon(y) = 1, & S(x) &= x, & S(y) &= y, \\ \Delta(z) &= \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z), & \varepsilon(z) &= 1, & S(z) &= z. \end{aligned}$$

The automorphism group of H_8 is the Klein four-group [SV12]. These automorphisms are given in Table 1, which are going to be used in Corollary 5.3.

Denote a set of central orthogonal idempotents of H_8 as

$$\begin{aligned} e_1 &= \frac{1}{8}(1 + x)(1 + y)(1 + z), & e_2 &= \frac{1}{8}(1 + x)(1 + y)(1 - z), \\ e_3 &= \frac{1}{8}(1 - x)(1 - y)(1 + iz), & e_4 &= \frac{1}{8}(1 - x)(1 - y)(1 - iz), \end{aligned}$$

$$e_5 = \frac{1-xy}{2}, \quad e_j e_k = \delta_{jk}, \quad j, k = 1, \dots, 5; i = \sqrt{-1}.$$

And denote idempotents $e'_5 = \frac{1}{4}(1-xy)(1+z)$, $e''_5 = \frac{1}{4}(1-xy)(1-z)$, then

$$\begin{aligned} H_8 &= H_8 e_1 \oplus H_8 e_2 \oplus H_8 e_3 \oplus H_8 e_4 \oplus H_8 e_5 \\ &= H_8 e_1 \oplus H_8 e_2 \oplus H_8 e_3 \oplus H_8 e_4 \oplus (H_8 e'_5 + H_8 e''_5), \end{aligned}$$

where $H_8 e'_5 \simeq H_8 e''_5$ as left H_8 -module, via $e'_5 \mapsto x e''_5, x e'_5 \mapsto e''_5$.

Definition 3.1. Denote $V_1(b) := \mathbb{K}\{v | x \cdot v = b^2 v, y \cdot v = b^2 v, z \cdot v = b v, b \in \{\pm 1, \pm i\}\}$, where v is a vector. Let $V_2 \simeq H_8 e'_5$ as left H_8 -module, the actions of the generators are given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 3.2. All simple left modules of H_8 are classified by $V_1(b), V_2, b \in \{\pm 1, \pm i\}$.

Remark 3.3. Thanks to the referee's reminder, the result was also obtained in [EA03] under a different basis.

In the remaining part of the article, $V_1(b)$ and V_2 always mean a simple left H_8 -module.

Lemma 3.4. ([Rad03, Proposition 2]) Let H is a bialgebra over field \mathbb{K} and suppose S is the antipode of H .

(1) If $L \in {}_H \mathcal{M}$, then $L \otimes H \in {}_H \mathcal{YD}^H$, the module and comodule actions are given by

$$h \cdot (\ell \otimes a) = h_{(2)} \cdot \ell \otimes h_{(3)} a S^{-1}(h_{(1)}), \quad \rho(\ell \otimes h) = (\ell \otimes h_{(1)}) \otimes h_{(2)}, \forall h, a \in H, \ell \in L.$$

Let $M \in {}_H \mathcal{YD}^H$.

(2) Suppose that $L \in {}_H \mathcal{M}$ and $p : M \longrightarrow L$ is a map of left H -modules. Then the linear map $f : M \longrightarrow L \otimes H$ defined by $f(m) = p(m_{(0)}) \otimes m_{(1)}$ for all $m \in M$ is a map of Yetter-Drinfel'd H -modules, where $L \otimes H$ has the structure described in part (1). Furthermore $\ker f$ is the largest Yetter-Drinfel'd H -submodule, indeed the largest subcomodule, contained in $\ker p$.

(3) M is isomorphic to a Yetter-Drinfel'd submodule of some $L \otimes H$ described in above.

Similarly according to Radford's method, any simple left Yetter-Drinfel'd module over H_8 could be constructed by the submodule of tensor product of a left module V of H_8 and H_8 itself, where the module and comodule structures are given by :

$$(3.1) \quad h \cdot (\ell \boxtimes g) = (h_{(2)} \cdot \ell) \boxtimes h_{(1)} g S(h_{(3)}),$$

$$(3.2) \quad \rho(\ell \boxtimes h) = h_{(1)} \otimes (\ell \boxtimes h_{(2)}), \forall h, g \in H_8, \ell \in V.$$

Here we use \boxtimes instead of \otimes to avoid confusion by using too many symbols of the tensor product. We are going to construct all simple left Yetter-Drinfel'd modules over H_8 in this way. Keeping in mind that H_8 is semisimple, it's possibly being done. In fact, it is much easier than making use of the fact that ${}^H_H\mathcal{YD} \simeq {}_{\mathcal{D}(H_8^{cop})}\mathcal{M}$. The following is a list of useful formulae for looking for simple objects of ${}^{H_8}_{H_8}\mathcal{YD}$.

Lemma 3.5.

$$(3.3) \quad (\text{id}^{\otimes 2} \otimes S)\Delta^{(2)}(z) = \frac{1}{4}[(1+y)z \otimes z \otimes z(1+x) + (1-y)z \otimes xz \otimes z(1+x) +$$

$$(3.4) \quad + (1+y)z \otimes yz \otimes z(1-x) + (y-1)z \otimes xyz \otimes z(1-x)]$$

$$(3.5) \quad z_{(2)} \otimes z_{(1)}?S(z_{(3)}) = \frac{1}{4}[z \otimes (1+y)z?z(1+x) + xz \otimes (1-y)z?z(1+x) +$$

$$(3.6) \quad + yz \otimes (1+y)z?z(1-x) + xyz \otimes (1-y)z?(x-1)].$$

$$(3.7) \quad z_{(2)} \otimes z_{(1)}S(z_{(3)}) = \frac{1}{4}[z \otimes (1+x)(1+y) + xz \otimes (1+x)(1-y) +$$

$$(3.8) \quad + yz \otimes (1-x)(1+y) + xyz \otimes (1-x)(1-y)],$$

$$(3.9) \quad z_{(2)} \otimes z_{(1)}xS(z_{(3)}) = \frac{1}{4}[z \otimes (1+x)(1+y) + xz \otimes (1+x)(y-1) +$$

$$(3.10) \quad + yz \otimes (1-x)(1+y) + xyz \otimes (x-1)(1-y)],$$

$$(3.11) \quad z_{(2)} \otimes z_{(1)}yS(z_{(3)}) = \frac{1}{4}[z \otimes (1+x)(1+y) + xz \otimes (1+x)(1-y) +$$

$$(3.12) \quad + yz \otimes (x-1)(1+y) + xyz \otimes (x-1)(1-y)],$$

$$(3.13) \quad z_{(2)} \otimes z_{(1)}xyS(z_{(3)}) = \frac{1}{4}[z \otimes (1+x)(1+y) + xz \otimes (1+x)(y-1) +$$

$$(3.14) \quad + yz \otimes (x-1)(1+y) + xyz \otimes (1-x)(1-y)],$$

$$(3.15) \quad z_{(2)} \otimes z_{(1)}zS(z_{(3)}) = \frac{1}{2}[z \otimes (1+y)z + xyz \otimes x(y-1)z],$$

$$(3.16) \quad z_{(2)} \otimes z_{(1)}xzS(z_{(3)}) = \frac{1}{2}[z \otimes (1+y)z + xyz \otimes x(1-y)z],$$

$$(3.17) \quad z_{(2)} \otimes z_{(1)}yzS(z_{(3)}) = \frac{1}{2}[z \otimes x(1+y)z + xyz \otimes (y-1)z],$$

$$(3.18) \quad z_{(2)} \otimes z_{(1)}xyzS(z_{(3)}) = \frac{1}{2}[z \otimes x(1+y)z + xyz \otimes (1-y)z].$$

Definition 3.6. Define $M\langle b, g \rangle := \mathbb{K}\{v \boxtimes g | v \in V_1(b)\}$, where $b \in \{\pm 1, \pm i\}$ and $g \in \{1, x, y, xy\}$.

Lemma 3.7. *There are eight pairwise non-isomorphic one dimensional Yetter-Drinfel'd modules over H_8 as $M\langle b, g \rangle$ with $(b, g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}$. The actions and coactions are given by*

$$(3.19) \quad x \cdot (v \boxtimes g) = b^2(v \boxtimes g), \quad y \cdot (v \boxtimes g) = b^2(v \boxtimes g), \quad z \cdot (v \boxtimes g) = b(v \boxtimes g),$$

$$(3.20) \quad \rho(v \boxtimes g) = g \otimes (v \boxtimes g), \quad v \boxtimes g \in M\langle b, g \rangle, \quad v \in V_1(b).$$

Proof. Let $v \in V_1(b)$, then

$$(3.21) \quad z \cdot (v \boxtimes 1) \stackrel{(3.7)}{=} \frac{bv}{4} \boxtimes [1 + x + b^2(1 - x)][1 + y + b^2(1 - y)],$$

$$(3.22) \quad z \cdot (v \boxtimes xy) \stackrel{(3.13)}{=} \frac{bv}{4} \boxtimes [1 + x + b^2(x - 1)][1 + y + b^2(y - 1)],$$

$$(3.23) \quad z \cdot (v \boxtimes x) \stackrel{(3.9)}{=} \frac{bv}{4} \boxtimes [1 + x + b^2(1 - x)][1 + y + b^2(y - 1)],$$

$$(3.24) \quad z \cdot (v \boxtimes y) \stackrel{(3.11)}{=} \frac{bv}{4} \boxtimes [1 + x + b^2(x - 1)][1 + y + b^2(1 - y)].$$

so

$$(3.25) \quad z \cdot (v \boxtimes 1) = bv \boxtimes 1, \quad z \cdot (v \boxtimes xy) = bv \boxtimes xy, \text{ when } b = \pm 1;$$

$$(3.26) \quad z \cdot (v \boxtimes x) = bv \boxtimes x, \quad z \cdot (v \boxtimes y) = bv \boxtimes y, \text{ when } b = \pm i.$$

Now it's easy to see that $M\langle b, g \rangle$ defined in above is a one-dimensional Yetter-Drinfel'd modules by Radford's method and the eight one-dimensional Yetter-Drinfel'd modules are pairwise non-isomorphic by observations on their actions and coactions. \square

Definition 3.8. Let $(g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}$ and denote three vector spaces as

$$M\langle(1, xy)\rangle := \mathbb{K}\{v \boxtimes 1, v \boxtimes xy | v \in V_1(i)\},$$

$$M\langle(x, y)\rangle := \mathbb{K}\{v \boxtimes x, v \boxtimes y | v \in V_1(1)\},$$

$$M\langle(g_1, g_2)\rangle := \mathbb{K}\{(v_1 + v_2) \boxtimes g_1, (v_1 - v_2) \boxtimes g_2 | v_1, v_2 \in V_2\}.$$

Lemma 3.9. *There are six pairwise non-isomorphic two-dimensional simple Yetter-Drinfel'd modules over H_8 as below, where the action and coaction are given by formulae (3.1) and (3.2).*

(1) $M\langle(1, xy)\rangle$, the actions of generators on $(v \boxtimes 1, v \boxtimes xy)$ are given by

$$x \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(2) $M\langle(x, y)\rangle$, the actions of generators on $(v \boxtimes x, v \boxtimes y)$ are given by

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(3) $M\langle(g_1, g_2)\rangle$, where $(g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}$. the actions of generators on the row vector $((v_1 + v_2) \boxtimes g_1, (v_1 - v_2) \boxtimes g_2)$ are given by

$$x \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. For the coactions are easy to see, we can focus on their structures as left H_8 -modules. Part (1) and (2) of the lemma can be checked by formulae (3.21) to (3.24). Let $v_1, v_2 \in V_2$, then

$$(3.27) \quad z \cdot (v_1 \boxtimes 1) \stackrel{(3.7)}{=} \frac{1}{2}[v_1 \boxtimes (x+y) + v_2 \boxtimes (x-y)],$$

$$(3.28) \quad z \cdot (v_2 \boxtimes 1) \stackrel{(3.7)}{=} \frac{1}{2}[v_1 \boxtimes (-x+y) + v_2 \boxtimes (-x-y)],$$

$$(3.29) \quad z \cdot (v_1 \boxtimes xy) \stackrel{(3.13)}{=} \frac{1}{2}[v_1 \boxtimes (x+y) + v_2 \boxtimes (y-x)],$$

$$(3.30) \quad z \cdot (v_2 \boxtimes xy) \stackrel{(3.13)}{=} \frac{1}{2}[v_1 \boxtimes (x-y) + v_2 \boxtimes (-x-y)],$$

$$(3.31) \quad z \cdot (v_1 \boxtimes y) \stackrel{(3.11)}{=} \frac{1}{2}[v_1 \boxtimes (1+xy) + v_2 \boxtimes (1-xy)],$$

$$(3.32) \quad z \cdot (v_2 \boxtimes y) \stackrel{(3.11)}{=} \frac{1}{2}[v_1 \boxtimes (-1+xy) + v_2 \boxtimes (-1-xy)],$$

$$(3.33) \quad z \cdot (v_1 \boxtimes x) \stackrel{(3.9)}{=} \frac{1}{2}[v_1 \boxtimes (1+xy) + v_2 \boxtimes (-1+xy)],$$

$$(3.34) \quad z \cdot (v_2 \boxtimes x) \stackrel{(3.9)}{=} \frac{1}{2}[v_1 \boxtimes (1-xy) + v_2 \boxtimes (-1-xy)].$$

So we have

$$\begin{aligned} z \cdot [(v_1 + v_2) \boxtimes 1] &= (v_1 - v_2) \boxtimes y, & z \cdot [(v_1 - v_2) \boxtimes y] &= (v_1 + v_2) \boxtimes 1, \\ z \cdot [(v_1 + v_2) \boxtimes x] &= (v_1 - v_2) \boxtimes 1, & z \cdot [(v_1 - v_2) \boxtimes 1] &= (v_1 + v_2) \boxtimes x, \\ z \cdot [(v_1 + v_2) \boxtimes xy] &= (v_1 - v_2) \boxtimes x, & z \cdot [(v_1 - v_2) \boxtimes x] &= (v_1 + v_2) \boxtimes xy, \\ z \cdot [(v_1 + v_2) \boxtimes y] &= (v_1 - v_2) \boxtimes xy, & z \cdot [(v_1 - v_2) \boxtimes xy] &= (v_1 + v_2) \boxtimes y. \end{aligned}$$

Part (3) is immediate to check. The six two-dimensional Yetter-Drinfel'd modules are pairwise non-isomorphic since they are pairwise non-isomorphic as comodules. \square

Lemma 3.10. *Let $b_1, b_2 \in \{\pm 1\}$ and $v \in V_1(b_2)$ and denote*

$$(3.35) \quad w_1^{b_1, b_2} \triangleq v \boxtimes (1 + ib_1 y)z, \quad w_2^{b_1, b_2} \triangleq v \boxtimes x(1 - ib_1 y)z.$$

Then $W^{b_1, b_2} = \mathbb{K}w_1^{b_1, b_2} + \mathbb{K}w_2^{b_1, b_2}$ is a family of 4 pairwise non-isomorphic two dimensional simple Yetter-Drinfel'd modules over H_8 with the actions of generators on the row vector $(w_1^{b_1, b_2}, w_2^{b_1, b_2})$ and coactions given by

$$\begin{aligned} x \mapsto \begin{pmatrix} 0 & -ib_1 \\ ib_1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -ib_1 \\ ib_1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} \frac{(1-ib_1)b_2}{2} & \frac{(1-ib_1)b_2}{2} \\ -\frac{(1-ib_1)b_2}{2} & \frac{(1-ib_1)b_2}{2} \end{pmatrix}, \\ \rho(w_1^{b_1, b_2}) = \frac{(1+y)z}{2} \otimes w_1^{b_1, b_2} + \frac{(1-y)z}{2} \otimes w_2^{b_1, b_2}, \end{aligned}$$

$$\rho(w_2^{b_1, b_2}) = \frac{x(1+y)z}{2} \otimes w_2^{b_1, b_2} + \frac{x(1-y)z}{2} \otimes w_1^{b_1, b_2}.$$

Proof. It's straightforward by the definition of Yetter-drinfel'd module. When $b_2 \neq b'_2$, $W^{b_1, b_2} \neq W^{b_1, b'_2}$ since we will see that their braidings are different in Proposition 4.1. As explained in the following remark, W^{b_1, b_2} has another basis $\{p_1, p_2\}$ with $p_1 \in V_1(b_2)$ and $p_2 \in V_1(-b_1 b_2 i)$. So $W^{b_1, b_2} \neq W^{b'_1, b_2}$ if $b_1 \neq b'_1$. \square

Remark 3.11. (1) Let $M = \mathbb{K}\{v \boxtimes z, v \boxtimes xz, v \boxtimes yz, v \boxtimes xyz | v \in V_1(b)\}$, $b \in \{\pm 1\}$. z acts on elements of M as

$$\begin{aligned} z \cdot (v \boxtimes z) &\stackrel{(3.15)}{=} \frac{bv}{2} \boxtimes (1 - x + y + xy)z, & z \cdot (v \boxtimes xz) &\stackrel{(3.16)}{=} \frac{bv}{2} \boxtimes (1 + x + y - xy)z, \\ z \cdot (v \boxtimes yz) &\stackrel{(3.17)}{=} \frac{bv}{2} \boxtimes (-1 + x + y + xy)z, & z \cdot (v \boxtimes xyz) &\stackrel{(3.18)}{=} \frac{bv}{2} \boxtimes (1 + x - y + xy)z. \end{aligned}$$

Then $M \simeq W^{1, b} \oplus W^{-1, b}$ as Yetter-Drinfel'd modules over H_8 .

(2) Let $f_{jk} \triangleq \frac{1}{4}[1 + (-1)^j x][1 + (-1)^k y]$, $j, k = 0, 1$. Denote

$$p_1 = w_1^{b_1, b_2} + ib_1 w_2^{b_1, b_2}, \quad p_2 = w_1^{b_1, b_2} - ib_1 w_2^{b_1, b_2},$$

then $W^{b_1, b_2} = \mathbb{K}p_1 + \mathbb{K}p_2$ with the actions of generators on the row vector (p_1, p_2) and coactions given by

$$\begin{aligned} x &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} b_2 & 0 \\ 0 & -ib_1 b_2 \end{pmatrix}, \\ \rho(p_1) &= [f_{00} - ib_1 f_{11}]z \otimes p_1 + [f_{10} + ib_1 f_{01}]z \otimes p_2, \\ \rho(p_2) &= [f_{00} + ib_1 f_{11}]z \otimes p_2 + [f_{10} - ib_1 f_{01}]z \otimes p_1. \end{aligned}$$

According to [Mas95, Remark 2.14], H_8 is presented by generators x, y, w , where the expressions containing z are replaced by

$$(3.36) \quad w = \left(f_{00} + \sqrt{i} f_{10} + \frac{1}{\sqrt{i}} f_{01} + i f_{11} \right) z, \quad w^2 = 1,$$

$$(3.37) \quad wx = yw, \quad S(w) = \left(\frac{1+i}{2}x + \frac{1-i}{2}y \right) w,$$

$$(3.38) \quad \Delta(w) = \left(\frac{1}{2}(1+xy) \otimes 1 + \frac{1+i}{4}(1-xy) \otimes x + \frac{1-i}{4}(1-xy) \otimes y \right) (w \otimes w).$$

Let $a + 1 = \pm \sqrt{2}$, we define

$$\begin{aligned} w_1^{(1)} &\triangleq (v_1 + iav_2) \boxtimes \frac{i}{2} [(x+y) + \sqrt{i}(x-y)]w + (av_1 - iv_2) \boxtimes \frac{1}{2} [(x+y) - \sqrt{i}(x-y)]w, \\ w_2^{(1)} &\triangleq (v_1 + iav_2) \boxtimes \frac{i}{2} [(1+xy) + \sqrt{i}(1-xy)]w - (av_1 - iv_2) \boxtimes \frac{1}{2} [(1+xy) - \sqrt{i}(1-xy)]w, \end{aligned}$$

$$w_1^{(2)} \triangleq (v_1 - iav_2) \boxtimes \frac{i}{2} \left[(x+y) + \sqrt{i}(x-y) \right] w + (av_1 + iv_2) \boxtimes \frac{1}{2} \left[(x+y) - \sqrt{i}(x-y) \right] w,$$

$$w_2^{(2)} \triangleq (v_1 - iav_2) \boxtimes \frac{i}{2} \left[(1+xy) + \sqrt{i}(1-xy) \right] w - (av_1 + iv_2) \boxtimes \frac{1}{2} \left[(1+xy) - \sqrt{i}(1-xy) \right] w.$$

Lemma 3.12. *Let $a+1 = \pm\sqrt{2}$, there are 4 pairwise non-isomorphic simple Yetter-Drinfel'd modules W_1^a and W_2^a over H_8 as following*

- (1) *Let $W_1^a = \mathbb{K}w_1^{(1)} \oplus \mathbb{K}w_1^{(1)}$, then W_1^a is a two dimensional simple Yetter-Drinfel'd module over H_8 with actions given by*

$$\begin{cases} x \cdot w_1^{(1)} = -w_1^{(1)} \\ y \cdot w_1^{(1)} = w_1^{(1)} \\ z \cdot w_1^{(1)} = \frac{1}{2}(1-i)(a+1)w_2^{(1)} \\ w \cdot w_1^{(1)} = \frac{1}{2\sqrt{i}}(1-i)(a+1)w_2^{(1)} \end{cases} \quad \begin{cases} x \cdot w_2^{(1)} = w_2^{(1)} \\ y \cdot w_2^{(1)} = -w_2^{(1)} \\ z \cdot w_2^{(1)} = \frac{1}{2}(1+i)(a+1)w_1^{(1)} \\ w \cdot w_2^{(1)} = \frac{\sqrt{i}}{2}(1+i)(a+1)w_1^{(1)} \end{cases}$$

and coactions given by

$$\rho(w_1^{(1)}) = \frac{1}{2}(x+y)w \otimes w_1^{(1)} + \frac{\sqrt{i}}{2}(x-y)w \otimes w_2^{(1)},$$

$$\rho(w_2^{(1)}) = \frac{1}{2}(1+xy)w \otimes w_2^{(1)} + \frac{\sqrt{i}}{2}(1-xy)w \otimes w_1^{(1)}.$$

- (2) *Let $W_2^a = \mathbb{K}w_1^{(2)} \oplus \mathbb{K}w_1^{(2)}$, then W_2^a is a two dimensional simple Yetter-Drinfel'd module over H_8 with actions given by*

$$\begin{cases} x \cdot w_1^{(2)} = w_1^{(2)} \\ y \cdot w_1^{(2)} = -w_1^{(2)} \\ z \cdot w_1^{(2)} = \frac{1}{2}(1-i)(a+1)w_2^{(2)} \\ w \cdot w_1^{(2)} = \frac{\sqrt{i}}{2}(1-i)(a+1)w_2^{(2)} \end{cases} \quad \begin{cases} x \cdot w_2^{(2)} = -w_2^{(2)} \\ y \cdot w_2^{(2)} = w_2^{(2)} \\ z \cdot w_2^{(2)} = \frac{1}{2}(1+i)(a+1)w_1^{(2)} \\ w \cdot w_2^{(2)} = \frac{1}{2\sqrt{i}}(1+i)(a+1)w_1^{(2)} \end{cases}$$

and coactions given by

$$\rho(w_1^{(2)}) = \frac{1}{2}(x+y)w \otimes w_1^{(2)} + \frac{\sqrt{i}}{2}(x-y)w \otimes w_2^{(2)},$$

$$\rho(w_2^{(2)}) = \frac{1}{2}(1+xy)w \otimes w_2^{(2)} + \frac{\sqrt{i}}{2}(1-xy)w \otimes w_1^{(2)}.$$

Proof. It's straightforward to check by the definition of Yetter-drinfel'd module. Actually $M \simeq \bigoplus_{a+1=\pm\sqrt{2}} (W_1^a \oplus W_2^a)$ as Yetter-Drinfel'd modules over H_8 , where $M = \mathbb{K}\{v_j \boxtimes z, v_j \boxtimes xz, v_j \boxtimes yz, v_j \boxtimes xyz | v_j \in V_2, j = 1, 2\}$.

Since $\sqrt{i} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$, $\frac{1}{2} \sqrt{2} \sqrt{i}(1-i) = 1$. Denote $a+1 = b\sqrt{2}$, $b = \pm 1$, $p_1^{(1)} = \sqrt{i}w_1^{(1)} + w_2^{(1)}$, $p_2^{(1)} = -\sqrt{i}w_1^{(1)} + w_2^{(1)}$, then $W_1^a = \mathbb{K}p_1^{(1)} + \mathbb{K}p_2^{(1)}$ with actions on the

row vector $(p_1^{(1)}, p_2^{(1)})$ given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}.$$

Let $p_1^{(2)} = w_1^{(2)} + \frac{1}{\sqrt{i}}w_2^{(2)}$, $p_2^{(2)} = w_1^{(2)} - \frac{1}{\sqrt{i}}w_2^{(2)}$, then $W_2^a = \mathbb{K}p_1^{(2)} + \mathbb{K}p_2^{(2)}$ with actions on the row vector $(p_1^{(2)}, p_2^{(2)})$ given by

$$x \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}.$$

Now we can observe that $W_1^{-1+\sqrt{2}}$ is isomorphic to $W_1^{-1-\sqrt{2}}$ as modules (or comodules) under suitably chosen base, but they are not isomorphic as modules and comodules at the same time. So $W_1^{-1+\sqrt{2}} \neq W_1^{-1-\sqrt{2}}$ as Yetter-Drinfel'd modules. For the same reason, we have $W_2^{-1+\sqrt{2}} \neq W_2^{-1-\sqrt{2}}$ and $W_1^a \neq W_2^a$. \square

Obviously, any module in Lemma 3.9 is not isomorphic to any one of modules in Lemma 3.10 and 3.12 as comodules. As H_8 -modules, $W^{b_1, b_2} \simeq V_1(b_2) \oplus V_1(-b_1 b_2 i)$, and $W_1^a \simeq W_2^a \simeq V_2$. So Yetter-drinfel'd modules in Lemma 3.9, 3.10 and 3.12 are pairwise non-isomorphic. Keeping in mind that H_8 is semisimple, now we are arriving at

Theorem 3.13. *All the simple Yetter-Drinfel'd modules over H_8 are classified by*

- 8 pairwise non-isomorphic simple Yetter-drinfel'd modules of one-dimension:

$$M\langle b, g \rangle, \quad (b, g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}.$$

- 14 pairwise non-isomorphic simple Yetter-drinfel'd modules of two-dimension:

$$M\langle (1, xy) \rangle, M\langle (x, y) \rangle, M\langle (g_1, g_2) \rangle, W^{b_1, b_2}, W_1^a, W_2^a,$$

$$\text{where } (g_1, g_2) \in \{(1, y), (x, 1), (xy, x), (y, xy)\}, b_1, b_2 \in \{\pm 1\}, a + 1 = \pm \sqrt{2}.$$

Remark 3.14. Jun Hu and Yinhuo Zhang investigated $\mathcal{D}(H)$ -modules in [HZ07a] and [HZ07b] by using Radford's construction [Rad03], especially they constructed all simple modules of $\mathcal{D}(H_8)$ under a different basis comparing with ours.

4. NICHOLS ALGEBRAS IN ${}_{H_8}^{H_8}\mathcal{YD}$

In this section, we try to determine all the finite-dimensional Nichols algebras generated by Yetter-Drinfel'd modules over H_8 . As a byproduct, we calculate Gelfand-Kirillov dimensions for some Nichols algebras.

We begin by studying the Nichols algebras of simple Yetter-Drinfel'd modules.

$M \in {}_{H_8}^{H_8} \mathcal{YD}$	condition	$\dim \mathfrak{B}(M)$	$\text{GKdim } \mathfrak{B}(M)$
$M\langle b, g \rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	1
	$(b, g) \in \{(\pm i, x), (\pm i, y)\}$	2	0
$M\langle (1, xy) \rangle$		∞	2
$M\langle (x, y) \rangle$		∞	2
$M\langle (g_1, g_2) \rangle$	$(g_1, g_2) \in \{(1, y), (x, 1)\}$	∞	∞
	$(g_1, g_2) \in \{(xy, x), (y, xy)\}$	8	0
W^{b_1, b_2}	$b_1 = \pm 1, b_2 = -1$	8	0
	$b_1 = \pm 1, b_2 = 1$	∞	∞
W_1^a, W_2^a	$a + 1 = \pm \sqrt{2}$	∞	

TABLE 2. Nichols algebras of simple Yetter-Drinfel'd modules over H_8

Proposition 4.1. *Given a simple Yetter-Drinfel'd module M over H_8 , $\dim \mathfrak{B}(M)$ ($\text{GKdim } \mathfrak{B}(M)$ for some cases) is presented in Table 2. Especially,*

- (1) $\mathfrak{B}(M\langle b, g \rangle) = \begin{cases} \mathbb{K}[p], & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}, \\ \mathbb{K}[p]/(p^2) = \wedge \mathbb{K}p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\}. \end{cases}$
- (2) *The both braidings of $M\langle (g_1, g_2) \rangle$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$ and $W^{b_1, -1}$ for $b_1 = \pm 1$ are Cartan type A_2 , so their corresponding Nichols algebras are isomorphic to an algebra which is generated by p_1, p_2 satisfying relations $p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = 0, p_1^2 = p_2^2 = 0$.*

Proof. \diamond Because $c(p \otimes p) = g \cdot p \otimes p = \begin{cases} p \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ -p \otimes p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\} \end{cases}$ under the assumption that $M\langle b, g \rangle = \mathbb{K}p$, the part (1) is obvious.

\diamond As for the part (2), we only give a proof for the case $W^{b_1, -1}$ for $b_1 = \pm 1$. Let $p_1 = w_1^{b_1, b_2} + ib_1 w_2^{b_1, b_2}$ and $p_2 = w_1^{b_1, b_2} - ib_1 w_2^{b_1, b_2}$, then the braiding of W^{b_1, b_2} is given by

$$\begin{aligned} c(p_1 \otimes p_1) &= b_2 p_1 \otimes p_1, & c(p_2 \otimes p_2) &= b_2 p_2 \otimes p_2, \\ c(p_1 \otimes p_2) &= -b_2 p_2 \otimes p_1, & c(p_2 \otimes p_1) &= b_2 p_1 \otimes p_2. \end{aligned}$$

When $b_2 = 1$, $\text{GKdim } \mathfrak{B}(W^{b_1, 1}) = \infty$ according to [AAH16, Lemma 2.8]. When $b_2 = -1$, the braiding is type A_2 . As discussed in [AD05], the Nichols algebra $\mathfrak{B}(W^{b_1, -1})$ is generated by p_1, p_2 with relations $p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = 0, p_1^2 = p_2^2 = 0$. So $\dim(\mathfrak{B}(W^{b_1, -1})) = 8$.

- ◇ Let $p_1 = v \boxtimes 1, p_2 = v \boxtimes xy \in M\langle(1, xy)\rangle$, then $c(p_j \otimes p_k) = p_k \otimes p_j$, where $j, k = 1, 2$. If we view $M\langle(1, xy)\rangle = \mathbb{K}p_1 \oplus \mathbb{K}p_2$ as braided vector spaces, then $\text{GKdim } \mathfrak{B}(M\langle(1, xy)\rangle) = \text{GKdim } \mathfrak{B}(\mathbb{K}p_1) + \text{GKdim } \mathfrak{B}(\mathbb{K}p_2) = 2$ by Lemma 2.3. Similarly, $\text{GKdim } \mathfrak{B}(M\langle(x, y)\rangle) = 2$.
- ◇ Let $p_1 = (v_1 + v_2) \boxtimes 1, p_2 = (v_1 - v_2) \boxtimes y \in M\langle(1, y)\rangle$. The braiding is given by

$$\begin{aligned} c(p_1 \otimes p_1) &= p_1 \otimes p_1, & c(p_1 \otimes p_2) &= p_2 \otimes p_1, \\ c(p_2 \otimes p_1) &= -p_1 \otimes p_2, & c(p_2 \otimes p_2) &= p_2 \otimes p_2. \end{aligned}$$

By [AAH16, Lemma 2.8], $\text{GKdim } \mathfrak{B}(M\langle(1, y)\rangle) = \infty$. For the same reason, we obtain $\text{GKdim } \mathfrak{B}(M\langle(x, 1)\rangle) = \infty$.

- ◇ Let $\theta = \frac{1}{2}(i-1)(a+1)$, then

$$\begin{aligned} c(w_1^{(1)} \otimes w_1^{(1)}) &= -\theta w_2^{(1)} \otimes w_2^{(1)}, & c(w_1^{(1)} \otimes w_2^{(1)}) &= \theta w_1^{(1)} \otimes w_2^{(1)}, \\ c(w_2^{(1)} \otimes w_1^{(1)}) &= -\theta w_2^{(1)} \otimes w_1^{(1)}, & c(w_2^{(1)} \otimes w_2^{(1)}) &= \theta w_1^{(1)} \otimes w_1^{(1)}, \\ c(iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)}) &= -i\theta(iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)}), \\ c(-iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)}) &= i\theta(-iw_1^{(1)} \otimes w_1^{(1)} + w_2^{(1)} \otimes w_2^{(1)}), \end{aligned}$$

By induction,

$$\begin{aligned} \mathfrak{S}_{2n-1,1} \left((w_1^{(1)} \otimes w_2^{(1)})^{\otimes n} \right) &= \frac{(1+\theta)[1-(-\theta^2)^n]}{1+\theta^2} \left((w_1^{(1)} \otimes w_2^{(1)})^{\otimes n} \right), \\ \mathfrak{S}_{2n,1} \left((w_1^{(1)} \otimes w_2^{(1)})^{\otimes n} \otimes w_1^{(1)} \right) &= \frac{1-\theta+(-1)^n \theta^{2n+1}(1+\theta)}{1+\theta^2} \left((w_1^{(1)} \otimes w_2^{(1)})^{\otimes n} \otimes w_1^{(1)} \right). \end{aligned}$$

It means that $(w_1^{(1)} \otimes w_2^{(1)})^{\otimes n}$ is an eigenvector of \mathfrak{S}_{2n-1} and $(w_1^{(1)} \otimes w_2^{(1)})^{\otimes n} \otimes w_1^{(1)}$ is an eigenvector of \mathfrak{S}_{2n} both with nonzero eigenvalue. So $\dim \mathfrak{B}(W_1^a) = \infty$. And $\dim \mathfrak{B}(W_2^a) = \infty$ is similar to prove. □

Proposition 4.2. (1) $\mathfrak{B}[M\langle b, g \rangle \oplus M\langle b', g' \rangle] \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(M\langle b', g' \rangle)$ for $(b, g), (b', g') \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}$.

(2) When $(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$, then

$$\begin{aligned} \mathfrak{B}[M\langle b, g \rangle \oplus M\langle(1, xy)\rangle] &\simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(M\langle(1, xy)\rangle), \\ \mathfrak{B}[M\langle b, g \rangle \oplus M\langle(x, y)\rangle] &\simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(M\langle(x, y)\rangle). \end{aligned}$$

(3) $\mathfrak{B}[M\langle b, g \rangle \oplus M\langle(g_1, g_2)\rangle] \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(M\langle(g_1, g_2)\rangle)$ for the following cases

- (a) $(b, g) = (\pm i, x), (g_1, g_2) = (xy, x)$;
- (b) $(b, g) = (\pm i, y), (g_1, g_2) = (y, xy)$;
- (c) $(b, g) = (\pm 1, 1), (g_1, g_2) \in \{(xy, x), (y, xy)\}$.

- (4) $\mathfrak{B}[M\langle b, g \rangle \oplus W^{b_1, -1}] \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(W^{b_1, -1})$ for the following cases
- (a) $(b, g) \in \{(1, 1), (1, xy)\}$, $b_1 = \pm 1$;
 - (b) $(b, g) \in \{(i, x), (i, y)\}$, $b_1 = 1$;
 - (c) $(b, g) \in \{(-i, x), (-i, y)\}$, $b_1 = -1$.
- (5) $\mathfrak{B}[M\langle (xy, x) \rangle \oplus M\langle (y, xy) \rangle] \simeq \mathfrak{B}(M\langle (xy, x) \rangle) \otimes \mathfrak{B}(M\langle (y, xy) \rangle)$.
- (6) $\mathfrak{B}(W^{1, -1} \oplus W^{-1, -1}) \simeq \mathfrak{B}(W^{1, -1}) \otimes \mathfrak{B}(W^{-1, -1})$.
- (7) $\text{GKdim } \mathfrak{B}[M\langle b, g \rangle \oplus M\langle (g_1, g_2) \rangle] = \infty$ for $(b, g) = (\pm i, x)$, $(g_1, g_2) = (y, xy)$ or $(b, g) = (\pm i, y)$, $(g_1, g_2) = (xy, x)$.
- (8) $\text{GKdim } \mathfrak{B}[M\langle b, g \rangle \oplus W^{b_1, -1}] = \infty$ for $(b, g) \in \{(i, x), (i, y)\}$, $b_1 = -1$ or $(b, g) \in \{(-i, x), (-i, y)\}$, $b_1 = 1$.
- (9) $\dim \mathfrak{B}((M\langle (g_1, g_2) \rangle)^{\oplus 2}) = \infty$ for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$.
- (10) $\dim \mathfrak{B}(W^{b_1, -1} \oplus W^{b_1, -1}) = \infty$ for $b_1 = \pm 1$.

Remark 4.3. According to the above two proposition, we calculate some Nichols algebras over direct sum of two simple objects of ${}^{H_8}_{H_8} \mathcal{YD}$ in Table 3.

Proof. The part (1)-(6) are direct results of Lemma 2.3. We only prove some cases as a byproduct in the following.

- ◇ Let $p_1 = (v_1 + v_2) \boxtimes g_1$, $p_2 = (v_1 - v_2) \boxtimes g_2 \in M\langle (g_1, g_2) \rangle$, where $(g_1, g_2) \in \{(xy, x), (y, xy)\}$. Let $p = v \boxtimes g \in M\langle b, g \rangle$, then

$$c(p \otimes p_1) = \begin{cases} -p_1 \otimes p, & \text{if } g \in \{y, xy\} \\ p_1 \otimes p, & \text{if } g \in \{1, x\}, \end{cases} \quad c(p \otimes p_2) = \begin{cases} -p_2 \otimes p, & \text{if } g \in \{x, xy\} \\ p_2 \otimes p, & \text{if } g \in \{1, y\}, \end{cases}$$

$$c(p_1 \otimes p) = \begin{cases} b^2 p \otimes p_1, & \text{if } g_1 = y \\ p \otimes p_1, & \text{if } g_1 = xy, \end{cases} \quad c(p_2 \otimes p) = \begin{cases} b^2 p \otimes p_2, & \text{if } g_2 = x \\ p \otimes p_2, & \text{if } g_2 = xy. \end{cases}$$

- When $(g_1, g_2) = (y, xy)$ and $(b, g) = (\pm i, x)$, then

$$\begin{aligned} c(p \otimes p_1) &= p_1 \otimes p_1, & c(p \otimes p_2) &= -p_2 \otimes p, \\ c(p_1 \otimes p) &= -p \otimes p_1, & c(p_2 \otimes p) &= p \otimes p_2. \end{aligned}$$

The generalized Dynkin diagram is given by Figure 1. According to [Hec09], $\dim \mathfrak{B}[M\langle \pm i, x \rangle \oplus M\langle (y, xy) \rangle] = \infty$.

- When $(g_1, g_2) = (xy, x)$ and $(b, g) = (\pm i, y)$, the generalized Dynkin diagram associated to the braiding is given by Figure 1. According to [Hec09], $\dim \mathfrak{B}[M\langle \pm i, y \rangle \oplus M\langle (xy, x) \rangle] = \infty$. We finish the part (7).
- As for cases listed in the part (6), $\mathfrak{B}[M\langle b, g \rangle \oplus M\langle (g_1, g_2) \rangle] \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(M\langle (g_1, g_2) \rangle)$ by Lemma 2.3.

$M \in {}^{H_8}_{H_8} \mathcal{YD}$	condition	$\dim \mathfrak{B}(M)$	$\text{GKdim} \mathfrak{B}(M)$
$M\langle b_1, g_1 \rangle \oplus M\langle b_2, g_2 \rangle$	$(b_1, g_1), (b_2, g_2) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	2
	$(b_1, g_1) \in \{(\pm 1, 1), (\pm 1, xy)\}$ $(b_2, g_2) \in \{(\pm i, x), (\pm i, y)\}$	∞	1
	$(b_1, g_1), (b_2, g_2) \in \{(\pm i, x), (\pm i, y)\}$	4	0
$M\langle b, g \rangle \oplus M\langle (1, xy) \rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	3
$M\langle b, g \rangle \oplus M\langle (x, y) \rangle$	$(b, g) \in \{(\pm 1, 1), (\pm 1, xy)\}$	∞	3
$M\langle (g_1, g_2) \rangle \oplus M\langle (g'_1, g'_2) \rangle$	$(g_1, g_2) = (g'_1, g'_2) = (xy, x) \text{ or } (y, xy)$	∞	
	$(g_1, g_2) = (xy, x), (g'_1, g'_2) = (y, xy)$	64	0
$W^{b_1, -1} \oplus W^{b'_1, -1}$	$b_1 = b'_1 = \pm 1$	∞	
	$b_1 = 1, b'_1 = -1$	64	0
$M\langle (g_1, g_2) \rangle \oplus W^{b_1, -1}$	$(g_1, g_2) = (xy, x) \text{ or } (y, xy), b_1 = \pm 1$?	
$M\langle b, g \rangle \oplus M\langle (g_1, g_2) \rangle$	$(b, g) = (\pm i, x), (g_1, g_2) = (xy, x)$	16	0
	$(b, g) = (\pm i, x), (g_1, g_2) = (y, xy)$	∞	
	$(b, g) = (\pm i, y), (g_1, g_2) = (xy, x)$	∞	
	$(b, g) = (\pm i, y), (g_1, g_2) = (y, xy)$	16	0
	$(b, g) \in \{(\pm 1, 1)\}$ $(g_1, g_2) \in \{(xy, x), (y, xy)\}$	∞	1
$M\langle b, g \rangle \oplus W^{b_1, -1}$	$(b, g) \in \{(1, 1), (1, xy)\}, b_1 = \pm 1$	∞	1
	$(b, g) \in \{(i, x), (i, y)\}, b_1 = 1$	16	0
	$(b, g) \in \{(i, x), (i, y)\}, b_1 = -1$	∞	
	$(b, g) \in \{(-i, x), (-i, y)\}, b_1 = -1$	16	0
	$(b, g) \in \{(-i, x), (-i, y)\}, b_1 = 1$	∞	

TABLE 3. Nichols algebras over direct sum of two simple objects in ${}^{H_8}_{H_8} \mathcal{YD}$

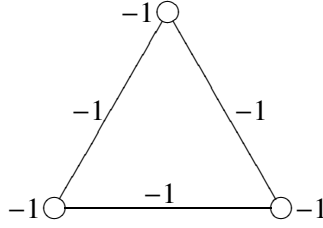


FIGURE 1.

◊ Let $p = v \boxtimes g \in M\langle b, g \rangle$, where $(b, g) \in \{(\pm 1, 1), (\pm 1, xy), (\pm i, x), (\pm i, y)\}$, then

$$\begin{aligned} c(p \otimes w_1^{b_1, -1}) &= \begin{cases} w_1^{b_1, -1} \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ ib_1 w_2^{b_1, -1} \otimes p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\}, \end{cases} \\ c(p \otimes w_2^{b_1, -1}) &= \begin{cases} w_2^{b_1, -1} \otimes p, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ -ib_1 w_1^{b_1, -1} \otimes p, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\}, \end{cases} \\ c(w_1^{b_1, -1} \otimes p) &= \begin{cases} bp \otimes w_1^{b_1, -1}, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ bp \otimes w_2^{b_1, -1}, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\}, \end{cases} \\ c(w_2^{b_1, -1} \otimes p) &= \begin{cases} bp \otimes w_2^{b_1, -1}, & \text{if } (b, g) \in \{(\pm 1, 1), (\pm 1, xy)\} \\ -bp \otimes w_1^{b_1, -1}, & \text{if } (b, g) \in \{(\pm i, x), (\pm i, y)\}. \end{cases} \end{aligned}$$

◦ In case $(b, g) \in \{(1, 1), (1, xy)\}$, according to Lemma 2.3, we have

$$\mathfrak{B}(M\langle b, g \rangle \oplus W^{b_1, -1}) \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(W^{b_1, -1}).$$

◦ In case $(b, g) \in \{(\pm i, x), (\pm i, y)\}$, if $ib_1 b = -1$, according to Lemma 2.3, we have $\mathfrak{B}(M\langle b, g \rangle \oplus W^{b_1, -1}) \simeq \mathfrak{B}(M\langle b, g \rangle) \otimes \mathfrak{B}(W^{b_1, -1})$. If $ib_1 b = 1$, the generalized Dynkin diagram associated to the braiding of $M\langle b, g \rangle \oplus W^{b_1, -1}$ is given by Figure 1. Now we finish the part (4) and (8).

◊ As for $(g_1, g_2) \in \{(xy, x), (y, xy)\}$, $\dim \mathfrak{B}((M\langle g_1, g_2 \rangle)^{\oplus 2}) = \infty$ by [Hec09], since the generalized Dynkin diagram associated to the braiding is given by Figure 2.

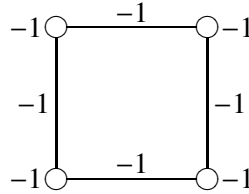


FIGURE 2.

◇ As for $W^{b_1, -1} \oplus W^{b'_1, -1}$ with b_1 and b'_1 in $\{\pm 1\}$. Let $p_1 = w_1^{b_1, -1} + ib'_1 w_2^{b_1, -1}$, $p_2 = w_1^{b_1, -1} - ib'_1 w_2^{b_1, -1}$, $p'_1 = w_1^{b'_1, -1} + ib'_1 w_2^{b'_1, -1}$ and $p'_2 = w_1^{b'_1, -1} - ib'_1 w_2^{b'_1, -1}$, then

$$\begin{aligned} c(p_1 \otimes p'_1) &= -p'_1 \otimes p_1, & c(p_2 \otimes p'_2) &= -p'_2 \otimes p_2, \\ c(p_1 \otimes p'_2) &= p'_2 \otimes p_1, & c(p_2 \otimes p'_1) &= -p'_1 \otimes p_2. \end{aligned}$$

When $b_1 = b'_1$, the generalized Dynkin diagram associated to the braiding is given by Figure 2. By [Hec09], $\dim \mathfrak{B}(W^{b_1, -1} \oplus W^{b_1, -1}) = \infty$. This finish (10).

When $b_1 = -b'_1$, then $p_2 = w_1^{b_1, -1} + ib_1 w_2^{b_1, -1}$, $p_1 = w_1^{b_1, -1} - ib_1 w_2^{b_1, -1}$, $p'_2 = w_1^{b'_1, -1} + ib_1 w_2^{b'_1, -1}$, $p'_1 = w_1^{b'_1, -1} - ib_1 w_2^{b'_1, -1}$, and

$$\begin{aligned} c(p'_2 \otimes p_2) &= -p_2 \otimes p'_2, & c(p'_1 \otimes p_1) &= -p_1 \otimes p'_1, \\ c(p'_2 \otimes p_1) &= p_1 \otimes p'_2, & c(p'_1 \otimes p_2) &= -p_2 \otimes p'_1. \end{aligned}$$

By Lemma 2.3, we have $\mathfrak{B}(W^{b_1, -1} \oplus W^{-b_1, -1}) \simeq \mathfrak{B}(W^{b_1, -1}) \otimes \mathfrak{B}(W^{-b_1, -1})$. This finish (6). □

Conjecture 4.4. $\dim \mathfrak{B}(M\langle(xy, x)\rangle \oplus W^{b_1, -1}) = \infty = \dim \mathfrak{B}(M\langle(y, xy)\rangle \oplus W^{b_1, -1})$ hold for $b_1 = \pm 1$.

Remark 4.5. Let $p_1 = (v_1 + v_2) \boxtimes xy$, $p_2 = (v_1 - v_2) \boxtimes x \in M\langle(xy, x)\rangle$, then

$$\begin{aligned} c(p_1 \otimes w_1^{b_1, -1}) &= w_1^{b_1, -1} \otimes p_1, & c(p_1 \otimes w_2^{b_1, -1}) &= w_2^{b_1, -1} \otimes p_1, \\ c(p_2 \otimes w_1^{b_1, -1}) &= ib_1 w_2^{b_1, -1} \otimes p_2, & c(p_2 \otimes w_2^{b_1, -1}) &= -ib_1 w_1^{b_1, -1} \otimes p_2, \\ c(w_1^{b_1, -1} \otimes p_1) &= p_2 \otimes w_1^{b_1, -1}, & c(w_1^{b_1, -1} \otimes p_2) &= p_1 \otimes w_2^{b_1, -1}, \\ c(w_2^{b_1, -1} \otimes p_1) &= -p_2 \otimes w_2^{b_1, -1}, & c(w_2^{b_1, -1} \otimes p_2) &= p_1 \otimes w_1^{b_1, -1}. \end{aligned}$$

Let $p'_1 = w_1^{b_1, -1} + ib_1 w_2^{b_1, -1}$ and $p'_2 = w_1^{b_1, -1} - ib_1 w_2^{b_1, -1}$, then

$$\begin{aligned} c(p_1 \otimes p'_1) &= p'_1 \otimes p_1, & c(p_1 \otimes p'_2) &= p'_2 \otimes p_1, \\ c(p_2 \otimes p'_1) &= p'_1 \otimes p_2, & c(p_2 \otimes p'_2) &= -p'_2 \otimes p_2, \\ c(p'_1 \otimes p_1) &= p_2 \otimes p'_2, & c(p'_1 \otimes p_2) &= ib_1 p_1 \otimes p'_2, \\ c(p'_2 \otimes p_1) &= p_2 \otimes p'_1, & c(p'_2 \otimes p_2) &= -ib_1 p_1 \otimes p'_1. \end{aligned}$$

Proof of Theorem A. Firstly, we recall the truth that for any submodule $M_1 \subset M_2 \in {}^H_H \mathcal{YD}$, $\mathfrak{B}(M_1) \subset \mathfrak{B}(M_2)$, $\dim \mathfrak{B}(M_2) = \infty$ if $\dim \mathfrak{B}(M_1) = \infty$. So the only possible $M \in {}^{H_8}_{H_8} \mathcal{YD}$ such that $\dim \mathfrak{B}(M) < \infty$ is in the list of Theorem A according to Table 2, Table 3, Proposition 4.1 and Proposition 4.2, under the assumption that Conjecture 4.4 is true. Now we only

need to check that Nichols algebras $\mathfrak{B}(M)$ for M listed in Theorem A is finite dimensional. In fact, $\Omega_1(n_1, n_2, n_3, n_4)$ is of Cartan type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2+n_3+n_4}$; $\Omega_k(n_1, n_2)$ for $k = 2, 3, 4, 5$ is of Cartan type $\underbrace{A_1 \times \cdots \times A_1}_{n_1+n_2} \times A_2$; Ω_k for $k = 6, 7$ is of Cartan type $A_2 \times A_2$.

5. HOPF ALGEBRAS OVER H_8

In this section, according to lifting method, we determine finite-dimensional Hopf algebra H with coradical H_8 such that its infinitesimal braiding is isomorphic to a Yetter-Drinfel'd module M listed in Theorem A. We begin by proving H is generated by elements of degree one in Theorem 5.1. That is, $\text{gr } H \simeq \mathfrak{B}(M) \# H_8$.

Theorem 5.1. *Let H be a finite-dimensional Hopf algebra over H_8 such that its infinitesimal braiding is isomorphic to a Yetter-Drinfel'd module over H_8 which is in the list of Theorem A. Then the diagram of H is a Nichols algebra, and consequently H is generated by the elements of degree one with respect to the coradical filtration.*

Proof. Since $\text{gr } H \simeq R \# H_8$, with $R = \bigoplus_{n \geq 0} R(n)$ the diagram of H , we need to prove that R is a Nichols algebra. Let $\mathcal{J} = \bigoplus_{n \geq 0} R(n)^*$ be the graded dual of R , then \mathcal{J} is a graded Hopf algebra in ${}^{H_8}_{H_8} \mathcal{YD}$ with $\mathcal{J}(0) = \mathbb{K}1$. According to [AS00, Lemma 5.5], $R(1) = \mathcal{P}(R)$ if and only if \mathcal{J} is generated as an algebra by $\mathcal{J}(1)$, that is, if \mathcal{J} is itself a Nichols algebra.

Considering $\mathfrak{B}(M) \in {}^{H_8}_{H_8} \mathcal{YD}$ for M in the list of Theorem A, since $\mathfrak{B}(M) = T(M)/\mathcal{I}$, in order to show $\mathcal{P}(\mathcal{J}) = \mathcal{J}(1)$, it is enough to prove that the relations that generate the ideal \mathcal{I} hold in \mathcal{J} . This can be done by a case by case computation. We perform here three cases, and leave the rest to the reader.

Suppose $M = \Omega_1(n_1, n_2, n_3, n_4)$. A direct computation shows that the elements r in \mathcal{J} representing the quadratic relations are primitive and since the braiding is -flips, they satisfy that $c(r \otimes r) = r \otimes r$. As $\dim \mathcal{J} < \infty$, it must be $r = 0$ in \mathcal{J} and hence there exists a projective algebra map $\mathfrak{B}(M) \rightarrow \mathcal{J}$, which implies that $\mathcal{P}(\mathcal{J}) = \mathcal{J}(1)$.

Suppose $M = \Omega_6$, then M is generated by elements $p_1 = (v_1 + v_2) \boxtimes xy$, $p_2 = (v_1 - v_2) \boxtimes x$, $p'_1 = (v_1 + v_2) \boxtimes y$, $p'_2 = (v_1 - v_2) \boxtimes xy$ and the ideal defining the Nichols algebra is generated by the elements $p_1^2, p_2^2, p_1'^2, p_2'^2, p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1, p_1' p_2' p_1' p_2' + p_2' p_1' p_2' p_1', p_1 p_1' + p_1' p_1, p_1 p_2' + p_2' p_1, p_2 p_1' - p_1' p_2, p_2 p_2' + p_2' p_2$. We can check directly that all those generators of the defining ideal of $\mathfrak{B}(M)$ are primitive elements, or by [Fan11, Theorem 6]. It's enough to show $c(r \otimes r) = r \otimes r$ for all generators given in above for the defining ideal. Since $\rho(p_1) = xy \otimes p_1, \rho(p_2) = x \otimes p_2, \rho(p'_1) = y \otimes p'_1$, so $\rho(p_1^2) = 1 \otimes p_1^2, \rho(p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1) =$

$1 \otimes (p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1), \rho(p_1 p'_1 + p'_1 p_1) = x \otimes (p_1 p'_1 + p'_1 p_1)$. It's easy to see $c(r \otimes r) = r \otimes r$ holds for $r = p_1^2, p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1$ and $p_1 p'_1 + p'_1 p_1$. We leave the rest to the reader.

Suppose $M = \Omega_4(n_1, n_2)$, then M is generated by elements $p_1 = w_1^{1,-1} + i w_2^{1,-1}, p_2 = w_1^{1,-1} - i w_2^{1,-1}, \{X_j\}_{j=1, \dots, n_1}, \{Y_k\}_{k=1, \dots, n_2}$ with $\mathbb{K}X_j \simeq M\langle i, x \rangle, \mathbb{K}Y_k \simeq M\langle i, y \rangle$ and the ideal defining the Nichols algebra is generated by the elements $p_1^2, p_2^2, p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1, X_j^2, \{X_{j_1} X_{j_2} + X_{j_2} X_{j_1}\}_{1 \leq j_1 < j_2 \leq n_1}, Y_k^2, \{Y_{k_1} Y_{k_2} + Y_{k_2} Y_{k_1}\}_{1 \leq k_1 < k_2 \leq n_2}, p_1 Y_k - Y_k p_1, p_2 Y_k + Y_k p_2, p_1 X_j - X_j p_1, p_2 X_j + X_j p_2$. We can check directly that all those generators of the defining ideal of $\mathfrak{B}(M)$ are primitive elements, or by [Fan11, Theorem 6]. It's enough to show $c(r \otimes r) = r \otimes r$ for all generators given in above for the defining ideal. Since $\rho(p_1) = (f_{00} - i f_{11})z \otimes p_1 + (f_{10} + i f_{01})z \otimes p_2, \rho(p_2) = (f_{00} + i f_{11})z \otimes p_2 + (f_{10} - i f_{01})z \otimes p_1, \rho(X_j) = x \otimes X_j,$

$$\begin{aligned} \rho(p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1) &= [(f_{00} - i f_{11})z(f_{00} + i f_{11})z]^2 \otimes p_1 p_2 p_1 p_2 + \\ &\quad + [(f_{00} + i f_{11})z(f_{00} - i f_{11})z]^2 \otimes p_2 p_1 p_2 p_1 + \\ &\quad + [(f_{10} + i f_{01})z(f_{10} - i f_{01})z]^2 \otimes p_2 p_1 p_2 p_1 + \\ &\quad + [(f_{10} - i f_{01})z(f_{10} + i f_{01})z]^2 \otimes p_1 p_2 p_1 p_2 \\ &= xy \otimes (p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1), \end{aligned}$$

$$\rho(p_1 X_j - X_j p_1) = (f_{00} + i f_{11})z \otimes (p_1 X_j - X_j p_1) + (f_{10} - i f_{01})z \otimes (p_2 X_j + X_j p_2).$$

Because

$$\begin{aligned} (f_{10} - i f_{01})z \cdot (p_1 X_j - X_j p_1) &= \frac{f_{10} - i f_{01}}{2} \cdot \left[((1+y)z \cdot p_1)(z \cdot X_j) - ((1-y)z \cdot X_j)(xz \cdot p_1) \right] \\ &= (-i)(f_{10} - i f_{01}) \cdot (p_1 X_j - X_j p_1) = 0, \end{aligned}$$

$$(f_{00} + i f_{11})z \cdot (p_1 X_j - X_j p_1) = (-i)(f_{00} + i f_{11}) \cdot (p_1 X_j - X_j p_1) = p_1 X_j - X_j p_1,$$

$$xy \cdot (p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1) = p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1,$$

$c(r \otimes r) = r \otimes r$ holds for $r = p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1$ and $p_1 X_j - X_j p_1$. We leave the rest to the reader. \square

Lemma 5.2. [AS00, Lemma 6.1] *Let H be a Hopf algebra, $\psi : H \rightarrow H$ an automorphism of Hopf algebras, V, W Yetter-Drinfel'd modules over H .*

(1) *Let V^ψ be the same space underlying V but with action and coaction*

$$h \cdot_\psi v = \psi(h) \cdot v, \quad \rho^\psi(v) = (\psi^{-1} \otimes \text{id})\rho(v), \quad h \in H, v \in V.$$

Then V^ψ is also a Yetter-Drinfel'd module over H . If $T : V \rightarrow W$ is a morphism in ${}^H_H \mathcal{YD}$, then $T^\psi : V^\psi \rightarrow W^\psi$ also is. Moreover, the braiding $c : V^\psi \otimes W^\psi \rightarrow W^\psi \otimes V^\psi$ coincides with the braiding $c : V \otimes W \rightarrow W \otimes V$.

- (2) If R is an algebra (resp., a coalgebra, a Hopf algebra) in ${}^H_H\mathcal{YD}$, then R^ψ also is, with the same structural maps.
- (3) Let R be a Hopf algebra in ${}^H_H\mathcal{YD}$. Then the map $\Psi : R^\psi \# H \rightarrow R \# H$ given by $\Psi(r \# h) = r \# \psi(h)$ is an isomorphism of Hopf algebras.

Corollary 5.3. (1) $[M\langle bi, x \rangle]^{\tau_3} \simeq M\langle -bi, y \rangle$, $b = \pm 1$.

(2) $[M\langle (xy, x) \rangle]^{\tau_3} \simeq M\langle (y, xy) \rangle$, $(W^{b_1, -1})^{\tau_3} \simeq W^{-b_1, -1}$ with $b_1 = \pm 1$.

(3) $\mathfrak{B}(\Omega_2(n_1, n_2)) \# H_8 \simeq \mathfrak{B}(\Omega_3(n_2, n_1)) \# H_8$, $\mathfrak{B}(\Omega_4(n_1, n_2)) \# H_8 \simeq \mathfrak{B}(\Omega_5(n_2, n_1)) \# H_8$.

Definition 5.4. For $n_1, n_2, n_3, n_4 \in \mathbb{N}^{\geq 0}$ with $n_1 + n_2 + n_3 + n_4 \geq 1$, and a set I_1 of parameters $\lambda_{j,s}, \mu_{j,t}, \zeta_{k,s}, \theta_{k,t}$ in \mathbb{K} with $j = 1, \dots, n_1, k = 1, \dots, n_2, s = 1, \dots, n_3, t = 1, \dots, n_4$, denote by $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ the algebra generated by $x, y, z, \{X_j\}_{j=1, \dots, n_1}, \{Y_k\}_{k=1, \dots, n_2}, \{p_s\}_{s=1, \dots, n_3}, \{q_t\}_{t=1, \dots, n_4}$ satisfying the following relations:

$$(5.1) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.2) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.3) \quad xX_j = -X_jx, \quad yX_j = -X_jy, \quad zX_j = iX_jxz,$$

$$(5.4) \quad xY_k = -Y_kx, \quad yY_k = -Y_ky, \quad zY_k = -iY_kxz,$$

$$(5.5) \quad xp_s = -p_sx, \quad yp_s = -p_sy, \quad zp_s = ip_sxz,$$

$$(5.6) \quad xq_t = -q_tx, \quad yq_t = -q_ty, \quad zq_t = -iq_txz,$$

$$(5.7) \quad X_j^2 = 0, \quad Y_k^2 = 0, \quad p_s^2 = 0, \quad q_t^2 = 0,$$

$$(5.8) \quad X_{j_1}X_{j_2} + X_{j_2}X_{j_1} = 0, \quad j_1, j_2 \in \{1, \dots, n_1\},$$

$$(5.9) \quad Y_{k_1}Y_{k_2} + Y_{k_2}Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \dots, n_2\},$$

$$(5.10) \quad p_{s_1}p_{s_2} + p_{s_2}p_{s_1} = 0, \quad s_1, s_2 \in \{1, \dots, n_3\},$$

$$(5.11) \quad q_{t_1}q_{t_2} + q_{t_2}q_{t_1} = 0, \quad t_1, t_2 \in \{1, \dots, n_4\},$$

$$(5.12) \quad X_jY_k + Y_kX_j = 0, \quad X_jp_s + p_sX_j = \lambda_{j,s}(1 - xy),$$

$$(5.13) \quad X_jq_t + q_tX_j = \mu_{j,t}(1 - xy), \quad Y_kp_s + p_sY_k = \zeta_{k,s}(1 - xy),$$

$$(5.14) \quad Y_kq_t + q_tY_k = \theta_{k,t}(1 - xy), \quad p_sq_t + q_tp_s = 0.$$

It is a Hopf algebra with its structure determined by

$$(5.15) \quad \Delta(X_j) = X_j \otimes 1 + x \otimes X_j, \quad \Delta(Y_k) = Y_k \otimes 1 + x \otimes Y_k,$$

$$(5.16) \quad \Delta(p_s) = p_s \otimes 1 + y \otimes p_s, \quad \Delta(q_t) = q_t \otimes 1 + y \otimes q_t,$$

$$(5.17) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Remark 5.5. (1) In fact, $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1) \simeq [T(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8] / \mathcal{I}(I_1)$, where $\mathcal{I}(I_1)$ is a Hopf ideal generated by relations (5.7)–(5.14). Especially, when parameters in I_1 are all equal to zero, then $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1) \simeq \mathfrak{B}(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8$.

(2) We can observe that any element of $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ can be expressed by a linear sum of $\{X_1^{\alpha_1} \cdots X_{n_1}^{\alpha_{n_1}} Y_1^{\beta_1} \cdots Y_{n_2}^{\beta_{n_2}} P_1^{\gamma_1} \cdots P_{n_3}^{\gamma_{n_3}} Q_1^{\kappa_1} \cdots Q_{n_4}^{\kappa_{n_4}} x^c y^d z^e\}$ for all parameteres $\alpha_1, \dots, \alpha_{n_1}, \beta_1, \dots, \beta_{n_2}, \gamma_1, \dots, \gamma_{n_3}, \kappa_1, \dots, \kappa_{n_4}, c, d, e$ in $\{0, 1\}$. So $\mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$ is finite dimensional.

Proposition 5.6. *Let H be a finite-dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_1(n_1, n_2, n_3, n_4)$, then $H \simeq \mathfrak{A}_1(n_1, n_2, n_3, n_4; I_1)$.*

Proof. By Theorem 5.1, we have $\text{gr } H \simeq \mathfrak{B}(\Omega_1(n_1, n_2, n_3, n_4)) \# H_8$. We can suppose H is generated by elements x, y, z in H and

$$(5.18) \quad X_j = (v \boxtimes x) \# 1 \in M\langle i, x \rangle \# 1, \quad j = 1, \dots, n_1,$$

$$(5.19) \quad Y_k = (v \boxtimes x) \# 1 \in M\langle -i, x \rangle \# 1, \quad k = 1, \dots, n_2,$$

$$(5.20) \quad p_s = (v \boxtimes y) \# 1 \in M\langle i, y \rangle \# 1, \quad s = 1, \dots, n_3,$$

$$(5.21) \quad q_t = (v \boxtimes y) \# 1 \in M\langle -i, y \rangle \# 1, \quad t = 1, \dots, n_4.$$

Then it's easy to check that formulae listed in Definition 5.4 except (5.7)–(5.14) hold in H from the bosonization $\mathfrak{B}[\Omega(n_1, n_2, n_3, n_4)] \# H_8$. Let $p = (v \boxtimes g) \# 1 \in [M\langle b, g \rangle] \# 1$, $p' = (v' \boxtimes g') \# 1 \in [M\langle b', g' \rangle] \# 1$, then $\Delta(pp' + p'p) = (pp' + p'p) \otimes 1 + gg' \otimes (pp' + p'p)$, from which we obtain the lifting relations (5.7)–(5.14) are only possible for the given generators. In fact, those lifting relations (5.7)–(5.14) generate a Hopf ideal. \square

Remark 5.7. (1) Suppose $H_1(b)$ is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle b, x \rangle$, where $b = \pm i$, then $H_1(b) \simeq \mathfrak{B}[M\langle b, x \rangle] \# H_8$. Denote $p = (v \boxtimes x) \# 1 \in [M\langle b, x \rangle] \# 1$, then $H_1(b)$ is generated by H_8, p , and with the following relations.

$$p^2 = 0, \quad xp = -px, \quad yp = -py, \quad zp = bpxz, \quad \Delta(p) = p \otimes 1 + x \otimes p.$$

(2) Suppose $H_2(b)$ is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle b, y \rangle$, where $b = \pm i$, then $H_2(b) \simeq \mathfrak{B}[M\langle b, y \rangle] \# H_8$. Denote $p = (v \boxtimes y) \# 1 \in [M\langle b, y \rangle] \# 1$, then $H_2(b)$ is generated by H_8, p , and with the following relations.

$$p^2 = 0, \quad xp = -px, \quad yp = -py, \quad zp = bpxz, \quad \Delta(p) = p \otimes 1 + y \otimes p.$$

- (3) $H_1(\pm i)$ are exactly the two nonisomorphic nonpointed self-dual Hopf algebras of dimension 16 with coradical H_8 described by Călinescu, Dăscălescu, Masuoka and Menini in [CDMM04]. In fact, $H_1(b) \simeq H_2(-b)$ by Corollary 5.3.

Lemma 5.8. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle(y, xy)\rangle$, then $H \simeq \mathfrak{B}[M\langle(y, xy)\rangle] \# H_8$. Denote $p_1 = [(v_1 + v_2) \boxtimes y] \# 1 \in M\langle(y, xy)\rangle \#$ and $p_2 = [(v_1 - v_2) \boxtimes xy] \# 1 \in M\langle(y, xy)\rangle \# 1$, then H is generated by x, y, z, p_1, p_2 , which satisfy the following relations.*

$$(5.22) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.23) \quad xp_1 = p_1x, \quad yp_1 = -p_1y, \quad zp_1 = p_2z,$$

$$(5.24) \quad xp_2 = -p_2x, \quad yp_2 = p_2y, \quad zp_2 = p_1xz,$$

$$(5.25) \quad p_1^2 = 0, \quad p_2^2 = 0, \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0.$$

Its Hopf algebra structure is determined by

$$(5.26) \quad \Delta(p_1) = p_1 \otimes 1 + y \otimes p_1, \quad \Delta(p_2) = p_2 \otimes 1 + xy \otimes p_2,$$

$$(5.27) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Proof. By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}[M\langle(y, xy)\rangle] \# H_8$. It's straightforward to prove that p_1^2, p_2^2 and $p_1p_2p_1p_2 + p_2p_1p_2p_1$ are primitive elements, so $H \simeq \text{gr } H$. \square

Lemma 5.9. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $M\langle(xy, x)\rangle$, then $H \simeq \mathfrak{B}[M\langle(xy, x)\rangle] \# H_8$. Let $p_1 = [(v_1 + v_2) \boxtimes xy] \# 1$, $p_2 = [(v_1 - v_2) \boxtimes x] \# 1$ be a basis of $M\langle(xy, x)\rangle \# 1$, then H is generated by x, y, z, p_1, p_2 , which satisfy the following relations.*

$$(5.28) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.29) \quad xp_1 = p_1x, \quad yp_1 = -p_1y, \quad zp_1 = p_2z,$$

$$(5.30) \quad xp_2 = -p_2x, \quad yp_2 = p_2y, \quad zp_2 = p_1xz,$$

$$(5.31) \quad p_1^2 = 0, \quad p_2^2 = 0, \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0.$$

Its Hopf algebra structure is determined by

$$(5.32) \quad \Delta(p_1) = p_1 \otimes 1 + xy \otimes p_1, \quad \Delta(p_2) = p_2 \otimes 1 + x \otimes p_2,$$

$$(5.33) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Proof. By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}[M\langle(xy, x)\rangle]\#H_8$. It's straightforward to prove that p_1^2, p_2^2 and $p_1p_2p_1p_2 + p_2p_1p_2p_1$ are primitive elements, so $H \simeq \text{gr } H$. In fact, $\mathfrak{B}[M\langle(xy, x)\rangle]\#H_8$ is isomorphic to $\mathfrak{B}[M\langle(y, xy)\rangle]\#H_8$ by Corollary 5.3. \square

Proposition 5.10. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_2(n_1, n_2)$, then $H \simeq \mathfrak{B}[\Omega_2(n_1, n_2)]\#H_8$. Denote*

$$(5.34) \quad p_1 = [(v_1 + v_2) \boxtimes xy]\#1, \quad p_2 = [(v_1 - v_2) \boxtimes x]\#1, \quad v_1, v_2 \in V_2,$$

$$(5.35) \quad X_j = (v \boxtimes x)\#1, \quad v \in V_1(i), \quad j = 1, \dots, n_1,$$

$$(5.36) \quad Y_k = (v' \boxtimes x)\#1, \quad v' \in V_1(-i), \quad k = 1, \dots, n_2,$$

then H is generated by $x, y, z, p_1, p_2, \{X_j\}_{j=1, \dots, n_1}, \{Y_k\}_{k=1, \dots, n_2}$ satisfying the following relations.

$$(5.37) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.38) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.39) \quad xp_1 = p_1x, \quad yp_1 = -p_1y, \quad zp_1 = p_2z,$$

$$(5.40) \quad xp_2 = -p_2x, \quad yp_2 = p_2y, \quad zp_2 = p_1xz,$$

$$(5.41) \quad p_1^2 = 0, \quad p_2^2 = 0, \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0,$$

$$(5.42) \quad xX_j = -X_jx, \quad yX_j = -yX_j, \quad zX_j = iX_jxz,$$

$$(5.43) \quad xY_k = -Y_kx, \quad yY_k = -yY_k, \quad zY_k = -iY_kxz,$$

$$(5.44) \quad X_{j_1}X_{j_2} + X_{j_2}X_{j_1} = 0, \quad j_1, j_2 \in \{1, \dots, n_1\},$$

$$(5.45) \quad Y_{k_1}Y_{k_2} + Y_{k_2}Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \dots, n_2\},$$

$$(5.46) \quad X_j^2 = 0, \quad Y_k^2 = 0, \quad X_jY_k + Y_kX_j = 0,$$

$$(5.47) \quad p_2X_j + X_jp_2 = 0, \quad p_2Y_k + Y_kp_2 = 0,$$

$$(5.48) \quad p_1X_j - X_jp_1 = 0, \quad p_1Y_k - Y_kp_1 = 0.$$

Its Hopf algebra structure is determined by

$$(5.49) \quad \Delta(p_1) = p_1 \otimes 1 + xy \otimes p_1, \quad \Delta(p_2) = p_2 \otimes 1 + x \otimes p_2,$$

$$(5.50) \quad \Delta(X_j) = X_j \otimes 1 + x \otimes X_j, \quad \Delta(Y_k) = Y_k \otimes 1 + x \otimes Y_k,$$

$$(5.51) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Proof. By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}[\Omega_2(n_1, n_2)]\#H_8$. By Lemma 5.9 and Proposition 5.6, we only need to prove that the lifting relations (5.47) and (5.48) are only possible by the given

generators, which can be obtained from the following formulae

$$\begin{aligned}
x(p_1X_j - X_jp_1) &= -(p_1X_j - X_jp_1)x, & x(p_1Y_k - Y_kp_1) &= -(p_1Y_k - Y_kp_1), \\
\Delta(p_1X_j - X_jp_1) &= (p_1X_j - X_jp_1) \otimes 1 + y \otimes (p_1X_j - X_jp_1), \\
\Delta(p_1Y_k - Y_kp_1) &= (p_1Y_k - Y_kp_1) \otimes 1 + y \otimes (p_1Y_k - Y_kp_1), \\
\Delta(p_2X_j + X_jp_2) &= (p_2X_j + X_jp_2) \otimes 1 + 1 \otimes (p_2X_j + X_jp_2), \\
\Delta(p_2Y_k + Y_kp_2) &= (p_2Y_k + Y_kp_2) \otimes 1 + 1 \otimes (p_2Y_k + Y_kp_2).
\end{aligned}$$

So $H \simeq \text{gr } H$. □

Definition 5.11. For $\lambda \in \mathbb{K}$, denote by $\mathfrak{U}_6(\lambda)$ the algebra generated by $x, y, z, p_1, p_2, q_1, q_2$ satisfying the following relations

$$(5.52) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.53) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.54) \quad xp_1 = p_1x, \quad yp_1 = -p_1y, \quad zp_1 = p_2z,$$

$$(5.55) \quad xp_2 = -p_2x, \quad yp_2 = p_2y, \quad zp_2 = p_1xz,$$

$$(5.56) \quad p_1^2 = 0, \quad p_2^2 = 0, \quad p_1p_2p_1p_2 + p_2p_1p_2p_1 = 0,$$

$$(5.57) \quad xq_1 = q_1x, \quad yq_1 = -q_1y, \quad zq_1 = q_2z,$$

$$(5.58) \quad xq_2 = -q_2x, \quad yq_2 = q_2y, \quad zq_2 = q_1xz,$$

$$(5.59) \quad q_1^2 = 0, \quad q_2^2 = 0, \quad q_1q_2q_1q_2 + q_2q_1q_2q_1 = 0,$$

$$(5.60) \quad p_1q_1 + q_1p_1 = \lambda(1 - x), \quad p_2q_2 + q_2p_2 = \lambda(1 - y),$$

$$(5.61) \quad p_1q_2 - q_2p_1 = 0, \quad p_2q_1 + q_1p_2 = 0.$$

It is a Hopf algebra with its structure determined by

$$(5.62) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz],$$

$$(5.63) \quad \Delta(p_1) = p_1 \otimes 1 + y \otimes p_1, \quad \Delta(p_2) = p_2 \otimes 1 + xy \otimes p_2,$$

$$(5.64) \quad \Delta(q_1) = q_1 \otimes 1 + xy \otimes q_1, \quad \Delta(q_2) = q_2 \otimes 1 + x \otimes q_2.$$

Remark 5.12. In fact, $\mathfrak{U}_6(\lambda) \simeq [T(\Omega_6)\#H_8]/I(\lambda)$, where $I(\lambda)$ is a Hopf ideal generated by relations (5.56), (5.59), (5.60) and (5.61). It's obvious that $\mathfrak{U}_6(\lambda)$ is finite-dimensional.

Proposition 5.13. Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to Ω_6 , then $H \simeq \mathfrak{U}_6(\lambda)$.

Proof. By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}(\Omega_6)\#H_8$. We can suppose that H is generated by generators x, y, z in H_8 and $p_1 = [(v_1 + v_2) \boxtimes y]\#1$, $p_2 = [(v_1 - v_2) \boxtimes xy]\#1$, $q_1 = [(v_1 + v_2) \boxtimes xy]\#1$,

$q_2 = [(v_1 - v_2) \boxtimes x] \# 1$ in $[M\langle(y, xy)\rangle \oplus M\langle(xy, x)\rangle] \# 1$. It's easy to see that formulae above in Definition 5.11 except (5.60) and (5.61) hold in H from the bosonization $\mathfrak{B}(\Omega_6) \# H_8$ and Lemma 5.8, 5.9. Since $\text{gr}[T(\Omega_6) \# H_8]/\mathcal{I}(\lambda) \simeq \mathfrak{B}(\Omega_6) \# H_8$, it's enough to prove that (5.60) and (5.61) are the only possible lifting relations by the given generators.

Since $r = 0$ in $\text{gr } H$ for $r = p_1 q_1 + q_1 p_1, p_2 q_2 + q_2 p_2, p_1 q_2 - q_2 p_1, p_2 q_1 + q_1 p_2$, we have $r \in H_8 \oplus \mathbb{K}(\Omega_5 \# 1)$. It's only possible that $p_1 q_1 + q_1 p_1 = \lambda_1(1 - x)$, $p_2 q_2 + q_2 p_2 = \lambda_2(1 - y)$, $p_1 q_2 - q_2 p_1 = \lambda_3(1 - xy)$, $p_2 q_1 + q_1 p_2 = 0$ for λ_1, λ_2 and λ_3 in \mathbb{K} , because

$$\begin{aligned}\Delta(p_1 q_1 + q_1 p_1) &= (p_1 q_1 + q_1 p_1) \otimes 1 + x \otimes (p_1 q_1 + q_1 p_1), \\ \Delta(p_2 q_2 + q_2 p_2) &= (p_2 q_2 + q_2 p_2) \otimes 1 + y \otimes (p_2 q_2 + q_2 p_2), \\ \Delta(p_1 q_2 - q_2 p_1) &= (p_1 q_2 - q_2 p_1) \otimes 1 + xy \otimes (p_1 q_2 - q_2 p_1), \\ \Delta(p_2 q_1 + q_1 p_2) &= (p_2 q_1 + q_1 p_2) \otimes 1 + 1 \otimes (p_2 q_1 + q_1 p_2).\end{aligned}$$

Since $z(p_1 q_1 + q_1 p_1) = (p_2 q_2 + q_2 p_2)z$ and $z(p_1 q_2 - q_2 p_1) = (p_2 q_1 + q_1 p_2)xz$, we have $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$. \square

Lemma 5.14. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $W^{b_1, -1}$, where $b_1 = \pm 1$. Then there exist parameters λ_1 and λ_2 in \mathbb{K} such that H is generated by x, y, z, p_1, p_2 , which satisfy the following relations.*

$$(5.65) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.66) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.67) \quad xp_1 = p_1 x, \quad yp_1 = p_1 y, \quad xp_2 = -p_2 x, \quad yp_2 = -p_2 y,$$

$$(5.68) \quad zp_1 = -p_1 z, \quad zp_2 = ib_1 p_2 xz,$$

$$(5.69) \quad p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = ib_1 \lambda_1(1 - xy),$$

$$(5.70) \quad p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = \lambda_2(1 - xy).$$

Its Hopf algebra structure is determined by

$$(5.71) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz],$$

$$(5.72) \quad \Delta(p_1) = [f_{00} - ib_1 f_{11}]z \otimes p_1 + [f_{10} + ib_1 f_{01}]z \otimes p_2 + p_1 \otimes 1,$$

$$(5.73) \quad \Delta(p_2) = [f_{00} + ib_1 f_{11}]z \otimes p_2 + [f_{10} - ib_1 f_{01}]z \otimes p_1 + p_2 \otimes 1.$$

Remark 5.15. In fact, $H \simeq [T(W^{b_1, -1}) \# H_8]/\mathcal{I}(\lambda_1, \lambda_2)$, where $\mathcal{I}(\lambda_1, \lambda_2)$ is a Hopf ideal generated by (5.69) and (5.70). It's obvious that H is finite dimensional.

Proof. By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}(W^{b_1, -1}) \# H_8$. We can suppose H is generated by x, y, z and p_1, p_2 with x, y, z in H and $p_1 = (w_1^{b_1, -1} + ib_1 w_2^{b_1, -1}) \# 1$, $p_2 = (w_1^{b_1, -1} - ib_1 w_2^{b_1, -1}) \# 1$.

Formulae (5.67), (5.68), (5.72) and (5.73) hold in H by a straightforward computation for the bosonization $\mathfrak{B}(W^{b_1, -1}) \# H_8$. Since

$$\begin{aligned}\Delta(p_1^2) &= \frac{1}{2}(1 + xy) \otimes p_1^2 + \frac{ib_1}{2}(1 - xy) \otimes p_2^2 + p_1^2 \otimes 1, \\ \Delta(p_2^2) &= \frac{1}{2}(1 + xy) \otimes p_2^2 - \frac{ib_1}{2}(1 - xy) \otimes p_1^2 + p_2^2 \otimes 1,\end{aligned}$$

there must exist a parameter $\lambda_1 \in \mathbb{K}$ such that $p_1^2 = \lambda_1(1 - xy)$ and $p_2^2 = ib_1 \lambda_1(1 - xy)$.

$$\begin{aligned}\Delta(p_1 p_2) &= \frac{1}{2}(x + y) \otimes p_1 p_2 + \frac{ib_1}{2}(x - y) \otimes p_2 p_1 + p_1 p_2 \otimes 1 + \\ &\quad + p_2 [f_{00} - ib_1 f_{11}] z \otimes p_1 + p_2 [f_{10} + ib_1 f_{01}] z \otimes p_2 + \\ &\quad + p_1 [f_{00} + ib_1 f_{11}] z \otimes p_2 + p_1 [f_{10} - ib_1 f_{01}] z \otimes p_1, \\ \Delta(p_2 p_1) &= \frac{1}{2}(x + y) \otimes p_2 p_1 + \frac{ib_1}{2}(y - x) \otimes p_1 p_2 + p_2 p_1 \otimes 1 + \\ &\quad + [f_{00} + ib_1 f_{11}] z p_1 \otimes p_2 + [f_{10} - ib_1 f_{01}] z p_1 \otimes p_1 + \\ &\quad + p_2 [f_{00} - ib_1 f_{11}] z \otimes p_1 + p_2 [f_{10} + ib_1 f_{01}] z \otimes p_2.\end{aligned}$$

Denote $\Delta(p_1 p_2) = B - A + E_1$ and $\Delta(p_2 p_1) = B + A + E_2$, where

$$\begin{aligned}A &= [f_{00} + ib_1 f_{11}] z p_1 \otimes p_2 + [f_{10} - ib_1 f_{01}] z p_1 \otimes p_1, \\ B &= p_2 [f_{00} - ib_1 f_{11}] z \otimes p_1 + p_2 [f_{10} + ib_1 f_{01}] z \otimes p_2, \\ E_2 &= \frac{1}{2}(x + y) \otimes p_2 p_1 + \frac{ib_1}{2}(y - x) \otimes p_1 p_2 + p_2 p_1 \otimes 1, \\ E_1 &= \frac{1}{2}(x + y) \otimes p_1 p_2 + \frac{ib_1}{2}(x - y) \otimes p_2 p_1 + p_1 p_2 \otimes 1.\end{aligned}$$

We can obtain $A^2 + B^2 = 0$, since

$$\begin{aligned}A^2 &= -\frac{1}{2}(x + y) p_1^2 \otimes p_2^2 + \frac{ib_1}{2}(1 - xy) p_1^2 \otimes p_1^2 = ib_1 \lambda_1^2 (1 - xy) \otimes (1 - xy), \\ B^2 &= -\frac{1}{2}(x + y) p_2^2 \otimes p_1^2 + \frac{ib_1}{2}(1 - xy) p_2^2 \otimes p_2^2 = -ib_1 \lambda_1^2 (1 - xy) \otimes (1 - xy).\end{aligned}$$

Keeping in mind that

$$\begin{aligned}p_1(p_1 p_2 + p_2 p_1) &= (p_2 p_1 + p_1 p_2) p_1, & p_2(p_1 p_2 + p_2 p_1) &= (p_2 p_1 + p_1 p_2) p_2, \\ p_1(p_1 p_2 - p_2 p_1) &= (p_2 p_1 - p_1 p_2) p_1, & p_2(p_1 p_2 - p_2 p_1) &= (p_2 p_1 - p_1 p_2) p_2, \\ (x + y) p_2 (f_{00} - ib_1 f_{11}) z &= -p_2 (f_{00} - ib_1 f_{11}) z (x + y),\end{aligned}$$

$$\begin{aligned}
(x-y)p_2(f_{00} - ib_1 f_{11})z &= p_2(f_{00} - ib_1 f_{11})z(x-y), \\
(x+y)p_2(f_{10} + ib_1 f_{01})z &= -p_2(f_{10} + ib_1 f_{01})z(x+y), \\
(x-y)p_2(f_{10} + ib_1 f_{01})z &= p_2(f_{10} + ib_1 f_{01})z(x-y), \\
(p_1 p_2 + p_2 p_1)p_2(f_{00} - ib_1 f_{11})z &= -p_2(f_{00} - ib_1 f_{11})z(p_1 p_2 + p_2 p_1), \\
(p_1 p_2 + p_2 p_1)p_2(f_{10} + ib_1 f_{01})z &= -p_2(f_{10} + ib_1 f_{01})z(p_1 p_2 + p_2 p_1),
\end{aligned}$$

we deduce $B(E_1 + E_2) + (E_1 + E_2)B = 0$. Similarly, we have $A(E_2 - E_1) + (E_2 - E_1)A = 0$.

$$\begin{aligned}
\Delta(p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1) &= (B - A + E_1)^2 + (B + A + E_2)^2 \\
&= 2(A^2 + B^2) + B(E_1 + E_2) + (E_1 + E_2)B + A(E_2 - E_1) + (E_2 - E_1)A + E_1^2 + E_2^2 \\
&= E_1^2 + E_2^2 = \left(\frac{1}{2}(x+y) \otimes p_1 p_2 + \frac{ib_1}{2}(x-y) \otimes p_2 p_1 + p_1 p_2 \otimes 1 \right)^2 + \\
&\quad + \left(\frac{1}{2}(x+y) \otimes p_2 p_1 + \frac{ib_1}{2}(y-x) \otimes p_1 p_2 + p_2 p_1 \otimes 1 \right)^2 \\
&= \frac{1}{2}(1+xy) \otimes (p_1 p_2)^2 - \frac{1}{2}(1-xy) \otimes (p_2 p_1)^2 + (p_1 p_2)^2 \otimes 1 + \\
&\quad + \frac{1}{2}(1+xy) \otimes (p_2 p_1)^2 - \frac{1}{2}(1-xy) \otimes (p_1 p_2)^2 + (p_2 p_1)^2 \otimes 1 \\
&= xy \otimes [(p_1 p_2)^2 + (p_2 p_1)^2] + [(p_1 p_2)^2 + (p_2 p_1)^2] \otimes 1.
\end{aligned}$$

So there exists a parameter $\lambda_2 \in \mathbb{K}$ such that $p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = \lambda_2(1 - xy)$.

We have $H \simeq [T(W^{b_1, -1}) \# H_8] / I(\lambda_1, \lambda_2)$, because $\text{gr} \left\{ [T(W^{b_1, -1}) \# H_8] / I(\lambda_1, \lambda_2) \right\} \simeq \mathfrak{B}(W^{b_1, -1}) \# H_8$. \square

Definition 5.16. For a set of parameters $I_7 = \{\lambda_j \in \mathbb{K} | j = 1, \dots, 5\}$, denote by $\mathfrak{A}_7(I_7)$ the algebra generated by $x, y, z, p_1, p_2, q_1, q_2$ satisfying the following relations

$$(5.74) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.75) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.76) \quad xp_1 = p_1x, \quad yp_1 = p_1y, \quad xp_2 = -p_2x, \quad yp_2 = -p_2y,$$

$$(5.77) \quad zp_1 = -p_1z, \quad zp_2 = ip_2xz,$$

$$(5.78) \quad p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = i\lambda_1(1 - xy),$$

$$(5.79) \quad p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = \lambda_2(1 - xy),$$

$$(5.80) \quad xq_1 = q_1x, \quad yq_1 = q_1y, \quad xq_2 = -q_2x, \quad yq_2 = -q_2y,$$

$$(5.81) \quad zq_1 = -q_1z, \quad zq_2 = -iq_2xz,$$

$$(5.82) \quad q_1^2 = \lambda_3(1 - xy), \quad q_2^2 = -i\lambda_3(1 - xy),$$

$$(5.83) \quad q_1 q_2 q_1 q_2 + q_2 q_1 q_2 q_1 = \lambda_4(1 - xy),$$

$$(5.84) \quad p_1 q_2 + q_2 p_1 = 0, \quad p_2 q_1 + q_1 p_2 = 0,$$

$$(5.85) \quad p_1 q_1 + q_1 p_1 = \lambda_5(x + y - 2), \quad p_2 q_2 + q_2 p_2 = -i\lambda_5(x - y).$$

It is a Hopf algebra with its structure determined by

$$(5.86) \quad \Delta(p_1) = [f_{00} - if_{11}]z \otimes p_1 + [f_{10} + if_{01}]z \otimes p_2 + p_1 \otimes 1,$$

$$(5.87) \quad \Delta(p_2) = [f_{00} + if_{11}]z \otimes p_2 + [f_{10} - if_{01}]z \otimes p_1 + p_2 \otimes 1,$$

$$(5.88) \quad \Delta(q_1) = [f_{00} + if_{11}]z \otimes q_1 + [f_{10} - if_{01}]z \otimes q_2 + q_1 \otimes 1,$$

$$(5.89) \quad \Delta(q_2) = [f_{00} - if_{11}]z \otimes q_2 + [f_{10} + if_{01}]z \otimes q_1 + q_2 \otimes 1,$$

$$(5.90) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Remark 5.17. In fact, $\mathfrak{A}_7(I_7) \simeq [T(W^{1,-1} \oplus W^{-1,-1}) \# H_8] / I(I_7)$, where $I(I_7)$ is a Hopf ideal generated by relations (5.78), (5.79), (5.82), (5.83), (5.84) and (5.85). Since

$$p_1 p_2 p_1 p_2 p_1 = p_1 [\lambda_2(1 - xy) - p_1 p_2 p_1 p_2] = [\lambda_2 p_1 - \lambda_1 p_2 p_1 p_2](1 - xy),$$

$$p_2 p_1 p_2 p_1 p_2 = p_2 [\lambda_2(1 - xy) - p_2 p_1 p_2 p_1] = [\lambda_2 p_2 - \lambda_1 i p_1 p_2 p_1](1 - xy),$$

$\dim \langle 1, p_1, p_2 \rangle < \infty$ for the subalgebra $\langle 1, p_1, p_2 \rangle$ generated by $1, p_1, p_2$. Similarly, we have $\dim \langle 1, q_1, q_2 \rangle < \infty$. We can deduce that

$$\dim A_7(I_7) = \dim \langle 1, p_1, p_2 \rangle \dim \langle 1, q_1, q_2 \rangle \dim H_8 < \infty.$$

Proposition 5.18. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to Ω_7 , then $H \simeq \mathfrak{A}_7(I_7)$.*

Proof. By Theorem 5.1, we have $\text{gr } H \simeq \mathfrak{B}(\Omega_7) \# H_8$. We can suppose H is generated by $x, y, z, p_1, p_2, q_1, q_2$ with $x, y, z \in H_8$ and

$$(5.91) \quad p_1 = (w_1^{1,-1} + iw_2^{1,-1}) \# 1, \quad p_2 = (w_1^{1,-1} - iw_2^{1,-1}) \# 1,$$

$$(5.92) \quad q_1 = (w_1^{-1,-1} - iw_2^{-1,-1}) \# 1, \quad q_2 = (w_1^{-1,-1} + iw_2^{-1,-1}) \# 1.$$

By lemma 5.14, we only need to prove that (5.84) and (5.85) hold in H . It's only possible for $p_1 q_2 + q_2 p_1 = 0, p_2 q_1 + q_1 p_2 = 0$, since $x(p_1 q_2 + q_2 p_1) = -(p_1 q_2 + q_2 p_1)x, x(p_2 q_1 + q_1 p_2) = -(p_2 q_1 + q_1 p_2)x$, and

$$\Delta(p_1 q_2 + q_2 p_1) = \frac{1}{2}[(1 + xy) + i(1 - xy)] \otimes (p_1 q_2 + q_2 p_1) + (p_1 q_2 + q_2 p_1) \otimes 1,$$

$$\Delta(p_2 q_1 + q_1 p_2) = \frac{1}{2}[(1 + xy) - i(1 - xy)] \otimes (p_2 q_1 + q_1 p_2) + (p_2 q_1 + q_1 p_2) \otimes 1,$$

Similarly, we can get (5.85), since

$$\begin{aligned} z(p_1 q_1 + q_1 p_1) &= (p_1 q_1 + q_1 p_1) z, \quad z(p_2 q_2 - q_2 p_2) = -(p_2 q_2 - q_2 p_2) z, \\ \Delta(p_1 q_1 + q_1 p_1) &= \frac{x+y}{2} \otimes (p_1 q_1 + q_1 p_1) + (p_1 q_1 + q_1 p_1) \otimes 1 + \frac{i(x-y)}{2} \otimes (p_2 q_2 - q_2 p_2), \\ \Delta(p_2 q_2 - q_2 p_2) &= \frac{x+y}{2} \otimes (p_2 q_2 - q_2 p_2) + (p_2 q_2 - q_2 p_2) \otimes 1 - \frac{i(x-y)}{2} \otimes (p_1 q_1 + q_1 p_1). \end{aligned}$$

We have $H \simeq \mathfrak{A}_7(I_7)$, because $\text{gr}\{[T(\Omega_7) \# H_8] / I(I_7)\} \simeq \mathfrak{B}(\Omega_7) \# H_8$. \square

Definition 5.19. For a set of parameters $I_4 = \{\lambda_1, \lambda_2, \lambda_{j,k} \in \mathbb{K} | j = 1, \dots, n_1, k = 1, \dots, n_2\}$, denote by $\mathfrak{A}_4(n_1, n_2; I_4)$ the algebra generated by $x, y, z, p_1, p_2, \{X_j\}_{j=1, \dots, n_1}, \{Y_k\}_{k=1, \dots, n_2}$ satisfying the following relations

$$(5.93) \quad x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy),$$

$$(5.94) \quad xy = yx, \quad zx = yz, \quad zy = xz,$$

$$(5.95) \quad xp_1 = p_1 x, \quad yp_1 = p_1 y, \quad xp_2 = -p_2 x, \quad yp_2 = -p_2 y,$$

$$(5.96) \quad zp_1 = -p_1 z, \quad zp_2 = ip_2 xz,$$

$$(5.97) \quad p_1^2 = \lambda_1(1 - xy), \quad p_2^2 = i\lambda_1(1 - xy),$$

$$(5.98) \quad p_1 p_2 p_1 p_2 + p_2 p_1 p_2 p_1 = \lambda_2(1 - xy),$$

$$(5.99) \quad xX_j = -X_j x, \quad yX_j = -X_j y, \quad zX_j = iX_j xz,$$

$$(5.100) \quad xY_k = -Y_k x, \quad yY_k = -Y_k y, \quad zY_k = iY_k xz,$$

$$(5.101) \quad X_{j_1}^2 = 0, \quad X_{j_1} X_{j_2} + X_{j_2} X_{j_1} = 0, \quad j_1, j_2 \in \{1, \dots, n_1\},$$

$$(5.102) \quad Y_{k_1}^2 = 0, \quad Y_{k_1} Y_{k_2} + Y_{k_2} Y_{k_1} = 0, \quad k_1, k_2 \in \{1, \dots, n_2\},$$

$$(5.103) \quad X_j Y_k + Y_k X_j = \lambda_{j,k}(1 - xy),$$

$$(5.104) \quad p_1 Y_k - Y_k p_1 = 0, \quad p_2 Y_k + Y_k p_2 = 0, \quad p_1 X_j - X_j p_1 = 0, \quad p_2 X_j + X_j p_2 = 0.$$

It is a Hopf algebra with its structure determined by

$$(5.105) \quad \Delta(X_j) = X_j \otimes 1 + x \otimes X_j, \quad \Delta(Y_k) = Y_k \otimes 1 + y \otimes Y_k,$$

$$(5.106) \quad \Delta(p_1) = (f_{00} - if_{11})z \otimes p_1 + (f_{10} + if_{01})z \otimes p_2 + p_1 \otimes 1,$$

$$(5.107) \quad \Delta(p_2) = (f_{00} + if_{11})z \otimes p_2 + (f_{10} - if_{01})z \otimes p_1 + p_2 \otimes 1,$$

$$(5.108) \quad \Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}[(1 + y)z \otimes z + (1 - y)z \otimes xz].$$

Remark 5.20. We can observe that

$$\dim \mathfrak{A}_4(n_1, n_2; I_4) = \dim \langle 1, p_1, p_2 \rangle \dim \langle 1, \{X_j\}_{j=1, \dots, n_1} \rangle \dim \langle 1, \{Y_k\}_{k=1, \dots, n_2} \rangle \dim H_8 < \infty$$

for subalgebra $\langle 1, p_1, p_2 \rangle$ generated by $1, p_1, p_2$, subalgebra $\langle 1, \{X_j\}_{j=1, \dots, n_1} \rangle$ generated by $1, \{X_j\}_{j=1, \dots, n_1}$, and subalgebra $\langle 1, \{Y_k\}_{k=1, \dots, n_2} \rangle$ generated by $1, \{Y_k\}_{k=1, \dots, n_2}$.

In fact, $\mathfrak{A}_4(n_1, n_2; I_4) \simeq T[\Omega_4(n_1, n_2)]\#H_8/\mathcal{I}(I_4)$, where $\mathcal{I}(I_4)$ is a Hopf ideal generated by relations (5.97), (5.98), (5.101), (5.102), (5.103), (5.104).

Proposition 5.21. *Suppose H is a finite dimensional Hopf algebra with coradical H_8 such that its infinitesimal braiding is isomorphic to $\Omega_4(n_1, n_2)$, then $H \simeq \mathfrak{A}_4(n_1, n_2; I_4)$.*

Proof. By Theorem 5.1, we have $\text{gr } H \simeq \mathfrak{B}[\Omega_4(n_1, n_2)]\#H_8$. We can suppose H is generated by $x, y, z, p_1, p_2, \{X_j\}_{j=1, \dots, n_1}, \{Y_k\}_{k=1, \dots, n_2}$ with $x, y, z \in H_8$ and

$$(5.109) \quad p_1 = (w_1^{b_1, -1} + ib_1 w_2^{b_1, -1})\#1, \quad p_2 = (w_1^{b_1, -1} - ib_1 w_2^{b_1, -1})\#1,$$

$$(5.110) \quad X_j = (v \boxtimes x)\#1, \quad j = 1, \dots, n_1,$$

$$(5.111) \quad Y_k = (v \boxtimes y)\#1, \quad v \in V_1(i), \quad k = 1, \dots, n_2.$$

As similarly proved in Proposition 5.6 and Lemma 5.14, formulae (5.93)–(5.103) and (5.105)–(5.108) hold in H . Since $r = 0$ in $\text{gr } H$ for $r = p_1 Y_k - Y_k p_1$ and $p_2 Y_k + Y_k p_2$, r is an element of at most degree one. It's only possible for

$$p_1 Y_k - Y_k p_1 = -\mu_k (-f_{10} + ib_1 f_{01})z, \quad p_2 Y_k + Y_k p_2 = -\mu_k (f_{00} - ib_1 f_{11})z + \mu_k 1,$$

because of the following relations

$$x(p_1 Y_k - Y_k p_1) = -(p_1 Y_k - Y_k p_1)x, \quad z(p_1 Y_k - Y_k p_1) = -b_1 i (p_1 Y_k - Y_k p_1)xz,$$

$$x(p_2 Y_k + Y_k p_2) = (p_2 Y_k + Y_k p_2)x, \quad z(p_2 Y_k + Y_k p_2) = (p_2 Y_k + Y_k p_2)z,$$

$$\begin{aligned} \Delta(p_1 Y_k - Y_k p_1) &= (p_1 Y_k - Y_k p_1) \otimes 1 + (f_{00} + ib_1 f_{11})z \otimes (p_1 Y_k - Y_k p_1) + \\ &\quad + (-f_{10} + ib_1 f_{01})z \otimes (p_2 Y_k + Y_k p_2), \end{aligned}$$

$$\begin{aligned} \Delta(p_2 Y_k + Y_k p_2) &= (p_2 Y_k + Y_k p_2) \otimes 1 + (f_{00} - ib_1 f_{11})z \otimes (p_2 Y_k + Y_k p_2) - \\ &\quad - (f_{10} + ib_1 f_{01})z \otimes (p_1 Y_k - Y_k p_1). \end{aligned}$$

Similarly, we get

$$p_1 X_j - X_j p_1 = -\mu'_j (f_{10} - ib_1 f_{01})z, \quad p_2 X_j + X_j p_2 = -\mu'_j (f_{00} - ib_1 f_{11})z + \mu'_j 1,$$

from the following formulae

$$x(p_1 X_j - X_j p_1) = -(p_1 X_j - X_j p_1)x, \quad z(p_1 X_j - X_j p_1) = -b_1 i (p_1 X_j - X_j p_1)xz,$$

$$x(p_2 X_j + X_j p_2) = (p_2 X_j + X_j p_2)x, \quad z(p_2 X_j + X_j p_2) = (p_2 X_j + X_j p_2)z,$$

$$\begin{aligned} \Delta(p_1 X_j - X_j p_1) &= (f_{00} + ib_1 f_{11})z \otimes (p_1 X_j - X_j p_1) + (p_1 X_j - X_j p_1) \otimes 1 + \\ &\quad + (f_{10} - ib_1 f_{01})z \otimes (p_2 X_j + X_j p_2), \end{aligned}$$

$$\begin{aligned} \Delta(p_2X_j + X_jp_2) &= (f_{00} - ib_1f_{11})z \otimes (p_2X_j + X_jp_2) + (p_2X_j + X_jp_2) \otimes 1 + \\ &\quad + (f_{10} + ib_1f_{01})z \otimes (p_1X_j - X_jp_1). \end{aligned}$$

Since $X_jY_k + Y_kX_j = \lambda_{j,k}(1 - xy)$, $p_1(X_jY_k + Y_kX_j) = (X_jY_k + Y_kX_j)p_1 \Rightarrow \mu'_j = \mu_k = 0$. So (5.104) holds in H . We have $H \simeq \mathfrak{A}_4(n_1, n_2; I_4)$ because $\text{gr } \{T[\Omega_4(n_1, n_2)]\#H_8/I(I_4)\} \simeq \mathfrak{B}[\Omega_4(n_1, n_2)]\#H_8$. \square

Proof of Theorem B. Let M be one of Yetter-Drinfel'd modules listed in Theorem A. We need to give a construction for any finite-dimensional Hopf algebra H over H_8 up to isomorphism such that its infinitesimal braiding is isomorphic to M . By Theorem 5.1, $\text{gr } H \simeq \mathfrak{B}(M)\#H_8$. According to Corollary 5.3, up to isomorphism, $\text{gr } H \simeq \mathfrak{B}(M)\#H_8$ for $M = \Omega_1(n_1, n_2, n_3, n_4)$, $\Omega_2(n_1, n_2)$, $\Omega_4(n_1, n_2)$, Ω_6 , Ω_7 . Proposition 5.6, 5.10, 5.21, 5.13 and 5.18 finish the proof.

ACKNOWLEDGEMENTS

I am indebted to my doctoral supervisor Prof. Naihong Hu for his encouragement and useful discussions which push me to consider the subject of finite dimensional Nichols algebras. I would like to thank my postdoctoral supervisor Prof. Wenxue Huang, Yunnan Li and Rongchuan Xiong for useful discussions and constructive comments. Finally I would like to thank the referees for careful reading and useful comments. This work was partially supported by the doctoral research project of Guangdong Province (gdbsh2014004).

REFERENCES

- [AAH16] N. Andruskiewitsch, I. Angiono, and I. Heckenberger, *On finite gk -dimensional nichols algebras over abelian groups*, arXiv preprint arXiv:1606.02521 (2016).
- [AAI] N. Andruskiewitsch, I. Angiono, and A. G. Iglesias, *Liftings of Nichols algebras of diagonal type I. Cartan type A.*, arXiv:1509.01622, Int. Math. Res. Notices, to appear.
- [AAI⁺14] N. Andruskiewitsch, I. Angiono, A. G. Iglesias, A. Masuoka, and C. Vay, *Lifting via cocycle deformation*, J. Pure Appl. Algebra **218** (2014), no. 4, 684–703. MR 3133699
- [ACG15] N. Andruskiewitsch, G. Carnovale, and G. A. García, *Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type I. Non-semisimple classes in $\mathbf{PSL}_n(q)$* , J. Algebra **442** (2015), 36–65. MR 3395052
- [ACG16] ———, *Finite-dimensional pointed Hopf algebras over finite simple groups of Lie type II: unipotent classes in symplectic groups*, Commun. Contemp. Math. **18** (2016), no. 4, 1550053, 35. MR 3493214
- [AD05] N. Andruskiewitsch and S. Dăscălescu, *On finite quantum groups at -1* , Algebr. Represent. Theory **8** (2005), no. 1, 11–34. MR 2136919

- [AFGV10] N. Andruskiewitsch, F. Fantino, M. Graña, and L. Vendramin, *Pointed Hopf algebras over some sporadic simple groups*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 11-12, 605–608. MR 2652482
- [AFGV11] ———, *Finite-dimensional pointed Hopf algebras with alternating groups are trivial*, Ann. Mat. Pura Appl. (4) **190** (2011), no. 2, 225–245. MR 2786171
- [AHS10] N. Andruskiewitsch, I. Heckenberger, and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010), no. 6, 1493–1547. MR 2766176
- [AM16] G. Andres Garcia and J. Matheus Jury Giraldo, *On Hopf Algebras over quantum subgroups*, eprint arXiv:1605.03995 (2016).
- [And02] N. Andruskiewitsch, *About finite dimensional Hopf algebras*, Quantum symmetries in theoretical physics and mathematics (Bariloche, 2000), Contemp. Math., vol. 294, Amer. Math. Soc., Providence, RI, 2002, pp. 1–57. MR 1907185 (2003f:16059)
- [AS98] N. Andruskiewitsch and H.-J. Schneider, *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3* , J. Algebra **209** (1998), no. 2, 658–691. MR 1659895
- [AS00] N. Andruskiewitsch and H.-J. Schneider, *Finite quantum groups and Cartan matrices*, Adv. Math. **154** (2000), no. 1, 1–45. MR 1780094
- [AS02] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras*, New directions in Hopf algebras, Math. Sci. Res. Inst. Publ., vol. 43, Cambridge Univ. Press, Cambridge, 2002, pp. 1–68. MR 1913436
- [AS10] ———, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. of Math. (2) **171** (2010), no. 1, 375–417. MR 2630042 (2011j:16058)
- [AV11] N. Andruskiewitsch and C. Vay, *Finite dimensional Hopf algebras over the dual group algebra of the symmetric group in three letters*, Comm. Algebra **39** (2011), no. 12, 4507–4517. MR 2863448
- [BZ10] K. A. Brown and J. J. Zhang, *Prime regular Hopf algebras of GK-dimension one*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 1, 260–302. MR 2661247
- [CDMM04] C. Călinescu, S. Dăscălescu, A. Masuoka, and C. Menini, *Quantum lines over non-cocommutative cosemisimple Hopf algebras*, J. Algebra **273** (2004), no. 2, 753–779. MR 2037722
- [EA03] A. El Alaoui, *The character table for a Hopf algebra arising from the Drinfel'd double*, J. Algebra **265** (2003), no. 2, 478–495. MR 1987013
- [FAGM16] F. Fantino, G. A. Garcia, and M. Mastnak, *On finite-dimensional copointed Hopf algebras over dihedral groups*, eprint arXiv:1608.06167 (2016).
- [Fan11] Xin Fang, *On defining ideals and differential algebras of Nichols algebras*, J. Algebra **346** (2011), 299–331. MR 2842083
- [FG11] F. Fantino and G. A. Garcia, *On pointed Hopf algebras over dihedral groups*, Pacific J. Math. **252** (2011), no. 1, 69–91. MR 2862142
- [FGV10] S. Freyre, M. Graña, and L. Vendramin, *On Nichols algebras over $\mathrm{PGL}(2, q)$ and $\mathrm{PSL}(2, q)$* , J. Algebra Appl. **9** (2010), no. 2, 195–208. MR 2646659 (2011g:16058)
- [GGI11] G. A. García and A. G. Iglesias, *Finite-dimensional pointed Hopf algebras over \mathfrak{S}_4* , Israel J. Math. **183** (2011), 417–444. MR 2811166
- [GHV11] M. Graña, I. Heckenberger, and L. Vendramin, *Nichols algebras of group type with many quadratic relations*, Adv. Math. **227** (2011), no. 5, 1956–1989. MR 2803792 (2012f:16077)
- [GIV14] A. G. Iglesias and C. Vay, *Finite-dimensional pointed or copointed Hopf algebras over affine racks*, J. Algebra **397** (2014), 379–406. MR 3119229

- [Gn00] M. Graña, *A freeness theorem for Nichols algebras*, J. Algebra **231** (2000), no. 1, 235–257. MR 1779599
- [Hec06] I. Heckenberger, *The Weyl groupoid of a Nichols algebra of diagonal type*, Invent. Math. **164** (2006), no. 1, 175–188. MR 2207786
- [Hec09] ———, *Classification of arithmetic root systems*, Adv. Math. **220** (2009), no. 1, 59–124. MR 2462836
- [HLV12] I. Heckenberger, A. Lochmann, and L. Vendramin, *Braided racks, Hurwitz actions and Nichols algebras with many cubic relations*, Transform. Groups **17** (2012), no. 1, 157–194. MR 2891215
- [HS10] I. Heckenberger and H.-J. Schneider, *Nichols algebras over groups with finite root system of rank two I*, J. Algebra **324** (2010), no. 11, 3090–3114. MR 2732989
- [HV14] I. Heckenberger and L. Vendramin, *Nichols algebras over groups with finite root system of rank two II*, J. Group Theory **17** (2014), no. 6, 1009–1034. MR 3276225
- [HV15] I. Heckenberger and L. Vendramin, *Nichols algebras over groups with finite root system of rank two III*, J. Algebra **422** (2015), 223–256. MR 3272075
- [HZ07a] Jun Hu and Yinhao Zhang, *The β -character algebra and a commuting pair in Hopf algebras*, Algebr. Represent. Theory **10** (2007), no. 5, 497–516. MR 2336009
- [HZ07b] ———, *Induced modules of semisimple Hopf algebras*, Algebra Colloq. **14** (2007), no. 4, 571–584. MR 2352888
- [Kap75] I. Kaplansky, *Bialgebras*, Lecture Notes in Mathematics, Department of Mathematics, University of Chicago, Chicago, Ill., 1975. MR 0435126 (55 #8087)
- [KL00] G. R. Krause and T. H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, revised ed., Graduate Studies in Mathematics, vol. 22, American Mathematical Society, Providence, RI, 2000. MR 1721834
- [KP66] G. I. Kac and V. G. Paljutkin, *Finite ring groups*, Trudy Moskov. Mat. Obšč. **15** (1966), 224–261. MR 0208401
- [Mas95] A. Masuoka, *Semisimple Hopf algebras of dimension 6, 8*, Israel J. Math. **92** (1995), no. 1-3, 361–373. MR 1357764
- [Mon93] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1993. MR 1243637
- [Nic78] W. D. Nichols, *Bialgebras of type one**, Communications in Algebra **6** (1978), no. 15, 1521–1552.
- [Rad85] D. E. Radford, *The structure of Hopf algebras with a projection*, J. Algebra **92** (1985), no. 2, 322–347. MR 778452 (86k:16004)
- [Rad03] ———, *On oriented quantum algebras derived from representations of the quantum double of a finite-dimensional Hopf algebra*, J. Algebra **270** (2003), no. 2, 670–695. MR 2019635
- [Sch96] P. Schauenburg, *A characterization of the Borel-like subalgebras of quantum enveloping algebras*, Comm. Algebra **24** (1996), no. 9, 2811–2823. MR 1396857
- [SV12] D. S. Sage and M. D. Vega, *Twisted Frobenius-Schur indicators for Hopf algebras*, J. Algebra **354** (2012), 136–147. MR 2879228
- [WLD16] Jinyong Wu, Gongxiang Liu, and Nanqing Ding, *Classification of affine prime regular Hopf algebras of GK-dimension one*, Adv. Math. **296** (2016), 1–54. MR 3490761
- [WZZ13] D.-G. Wang, J. J. Zhang, and G. Zhuang, *Hopf algebras of GK-dimension two with vanishing Ext-group*, J. Algebra **388** (2013), 219–247. MR 3061686

SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, GUANGZHOU UNIVERSITY, GUANGZHOU 510006, P.R. CHINA

E-mail address: blueponder@foxmail.com